Published by Institute of Physics Publishing for SISSA

Received: *January 31, 2007*
Accepted: *March 1, 2007*
Published: *March 28, 2007*

Supersymmetric wrapped membranes, AdS_2 spaces, and bubbling geometries

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Abstract: We perform a systematic study, in eleven dimensional supergravity, of the geometry of wrapped brane configurations admitting AdS_2 limits. Membranes wrapping holomorphic curves in Calabi-Yau manifolds are found to exhibit some novel features; in particular, for fourfolds or threefolds, the gravitational effect of the branes on the overall transverse space is only weakly restricted by the kinematics of the Killing spinor equation. We also study the AdS_2 limits of the wrapped brane supergravity descriptions. For membranes wrapped in a two-fold, we derive a set of AdS_2 supersymmetry conditions which upon analytic continuation coincide precisely with those for the half-BPS bubbling geometries of LLM. From membranes wrapped in a three-fold, we obtain a set of AdS_2 supersymmetry conditions which upon analytic continuation describe a class of spacetimes which we identify as quarter-BPS bubbling geometries in M-theory, with $SO(4) \times SO(3) \times U(1)$ isometry in Riemannian signature. We also study fivebranes wrapping a special lagrangian five-cycle in a fivefold, in the presence of membranes wrapping holomorphic curves, and employ the wrapped brane supersymmetry conditions to derive a classification of the general minimally supersymmetric AdS_2 geometry in M-theory.

Keywords: AdS-CFT Correspondence, Supergravity Models, Extended Supersymmetry.

http://jhep.sissa.it/archive/papers/jhep032007115/jhep032007115.pdf

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1. Introduction

The AdS/CFT correspondence [1] has provided an unparalleled theoretical laboratory in which ideas about quantum gravity, and their dual field theory manifestations, may be explored in a controlled mathematical framework. Though the best-understood examples of the duality are for supersymmetric four-dimensional field theories, and their AdS_5 duals in IIB string theory, the total space of AdS/CFT duals in string and M-theory is vast, and remains relatively unexplored, even for supersymmetric examples of the duality. This work is part of an ongoing project which aims to shed light on the general geometrical properties of supersymmetric AdS spacetimes in M-theory, the brane configurations that can give rise to them, and, by the correspondence, the associated general properties of all CFTs with M-theory duals.

The techniques we employ to mine the supergravity side of the correspondence involve, as a first step, extracting all the conditions restricting the geometry of supersymmetric brane configurations, and their associated AdS spaces, that are contained in the Killing spinor equation of eleven dimensional supergravity, and repackaging them in a geometrically transparent way. In doing so, one arrives at a set of necessary and sufficient conditions for a spacetime to admit the desired number of Killing spinors of the required form. A very useful way to package the information is to use the G-structures defined by the Killing spinors. This G-structure classification scheme was first formalised in the context of supergravity in [2]; it has since been applied in many contexts, including the classification of minimally supersymmetric solutions of lower [3]-[12] and eleven-dimensional [13, 14] supergravities. A refinement of this technique, developed for the classification of spacetimes with extended supersymmetry, has been given in [8, 15], and employed in eleven dimensions [15]-[20], and IIB [21]-[23].

In this paper, we will be concerned with the application of these classification techniques to wrapped brane configurations admitting AdS_2 limits. In keeping with the general philosophy of AdS/CFT, one would expect that every AdS spacetime in M-theory should arise as the decoupling limit of some brane configuration. Supersymmetric AdS spacetimes of different dimensionalities, preserving different ammounts of supersymmetry, may be obtained as the near-horizon limit of branes wrapped on supersymmetric cycles, for which a whole zoo of possibilities exist; for a review, see [24]. The central importance of wrapped brane configurations, and their AdS limits, to the AdS/CFT correspondence, means that these spacetimes have attracted a great deal of attention, using a variety of different approaches. Early investigations, following the work of Maldacena and Nuñez [25, 26], focussed on studying the near-horizon limit of wrapped brane spacetimes in certain special cases, which were accesible via a lower-dimensional gauged supergravity ansatz; see, for example, [27]-[30]. Wrapped brane configurations have been classified directly in eleven dimensional supergravity, under a variety of assumptions, by numerous different authors [31]-[37]. Seperately, various classes of supersymmetric AdS spacetimes in M-theory have been classified, by inserting an appropriately general AdS ansatz directly into the Killing spinor equation of eleven dimensional supergravity; minimally supersymmetric AdS_3 in [35], minimal purely magnetic AdS_4 in [38] and minimal AdS_5 in [39]. Some refinements of these AdS classifications have also appeared; for AdS_4 in [40] and AdS_5 with $\mathcal{N}=2$ supersymmetry in [41].

Recently, in [42], an explicit link between these different avenues of investigation was supplied, first by using general G-structure classification techniques to obtain the supergravity description of wrapped brane spacetimes, and then by systematically employing a procedure first used in [39] to take the AdS limits. Specifically, [42] was concerned with providing a supergravity description of M5 branes wrapped on supersymmetric cycles in manifolds of G_2, $SU(3)$ or $SU(2)$ holonomy, and then using this description to derive the supersymmetry conditions for the AdS limits of the wrapped brane configurations.

In this paper, we will apply the techniques of [42] to wrapped brane configurations in M-theory admitting AdS_2 limits. We will study fivebranes wrapped on special lagrangian (SLAG) five-cycles in Calabi-Yau fivefolds, in the presence of membranes wrapping holomorphic curves. We will also study membranes wrapping holomorphic curves (or, in alternatative teminology, Kähler two-cycles) in Calabi-Yau n-folds, $n=2,\ldots,5$. There is one remaining possibility for a cycle on which all spacelike dimensions of an M-brane can wrap: for fivebranes, the product of a SLAG-3 with a Kähler two-cycle in an $SU(3)\times SU(2)$ manifold, but we do not study this here.

From the supergravity description of fivebranes and membranes wrapped in a fivefold, we obtain a classification of all minimally supersymmetric AdS_2 geometries in M-theory. The supersymmetry conditions we obtain in this case are rather complicated. Our results for membranes wrapped on holomorphic curves in Calabi-Yau n-folds are more transparent. Before discussing them, it is worth reviewing the findings of [42] for fivebranes. For the probe brane analysis, one looks at the background with metric

$$\mathrm{d}s^2 = \mathrm{d}s^2(\mathbb{R}^{1,p}) + \mathrm{d}s^2(\mathcal{M}_{10-p-q}) + \mathrm{d}s^2(\mathbb{R}^q), \tag{1.1}$$

where $\mathcal{M}_{10-p-q}$ has G_2, $SU(3)$ or $SU(2)$ holonomy. One then introduces a probe fivebrane, extended along the $\mathbb{R}^{1,p}$ directions and wrapped on a cycle in $\mathcal{M}_{10-p-q}$. The kappa-symmetry projections for the probe, in every case, imply that although the supersymmetry is reduced by half, the structure group defined by the Killing spinors is preserved. This has a very important consequence in constraining the supergravity description of the system when backreaction is turned on - in particular, it means that the almost product structure of the spacetime, and the local flatness of the overall transverse space, is protected by supersymmetry up to warping. The input for the metric and Killing spinors in the derivation of the supergravity description is that the Killing spinors define the same algebraic structures as for the probe branes, are simultaneous eigenspinors of five independent projection operators, and that the metric contains a warped Minkowski factor of the appropriate dimensionality to represent the unwrapped brane worldvolume. The metric ansatz for the supergravity description is thus

$$\mathrm{d}s^2 = L^{-1}\mathrm{d}s^2(\mathbb{R}^{1,p}) + \mathrm{d}s^2(\mathcal{M}_{10-d}). \tag{1.2}$$

Nothing is assumed at all about the form of the metric on $\mathcal{M}_{10-d}$, beyond its independence (together with that of L) of the Minkowski coordinates. But then it is found that supersymmetry *implies* that the backreacted metric for the supergravity description admits an almost product structure, in other words, is of the form

$$\mathrm{d}s^2 = L^{-1}\mathrm{d}s^2(\mathbb{R}^{1,p}) + \mathrm{d}s^2(\mathcal{M}_{10-p-q}) + L^2\mathrm{d}s^2(\mathbb{R}^q), \tag{1.3}$$

where now $\mathcal{M}_{10-p-q}$ is deformed away from G-holonomy but still admits a G-structure. A point we wish to emphasise is that the gravitational effect of the fivebranes on the overall transverse space is restricted, by supersymmetry, to inducing a warping by L^2 (this warp factor will generically depend on the coordinates of the overall transverse space). So supersymmetry protects, up to warping, both the almost product structure of the spacetime and the flatness of the overall transverse space.

We have found the behaviour of membranes, derived under exactly the same assumptions, to be quite different. Membranes wrapping a cycle in a fivefold is a special case which does not illustrate these new features, since of course there are no overall transverse directions (though it does have the unusual property that membranes may be wrapped without breaking any supersymmetry). The new effects are most pronounced for membranes wrapping a holomorphic curve in a four-fold. The relevant background in this case is

$$\mathrm{d}s^2 = -dt^2 + \mathrm{d}s^2(\mathcal{M}_8) + \mathrm{d}s^2(\mathbb{R}^2), \tag{1.4}$$

where $\mathcal{M}_8$ has $SU(4)$ holonomy. As we shall see below from the probe brane analysis, the kappa-symmetry projection for a probe membrane, extended along the timelike direction and wrapping a holomorphic curve, breaks half the supersymmetry but now *increases* the structure group defined by the Killing spinors to $SU(5)$. Then, in deriving the supergravity description of the system, there is no symmetry principle which can protect the almost product structure of the spacetime, so, in contrast to what happens for fivebranes, the backreaction of the membranes on the overall transverse space is largely unconstrained. The Killing spinors preserved by a probe membrane wrapped in a four-fold define exactly the same algebraic structures as those for a probe membrane wrapped in a five-fold. Therefore there is no symmetry principle which allows us to distinguish the supergravity descriptions of the two systems, and no symmetry principle to confine the gravitational effects of membranes wrapped in a fourfold to deforming $\mathcal{M}_8$ away from $SU(4)$ holonomy and warping the overall transverse space. The backreaction is not completely unconstrained of course, since the spacetime must still admit an $SU(5)$ structure, satisfying certain conditions.

This effect is still present, though in reduced form, for membranes wrapped in a three-fold. In this case, the special holonomy of the background is $SU(3)$. The kappa-symmetry projection for a probe breaks half the supersymmetry, but again increases the structure group defined by the Killing spinors, this time to $SU(3) \times SU(2)$. In this case, the almost product structure of the spacetime is preserved. However, we shall see that it is consistent with the Killing spinor equation for the gravitational effect of the membranes on the overall transverse space to both induce a warping, and to deform the overall transverse space away from being flat to being of $SU(2)$ holonomy. We find that in the supergravity description, the metric is given by

$$\mathrm{d}s^2 = -\Delta^2\mathrm{d}t^2 + \mathrm{d}s^2(\mathcal{M}_{SU(3)}) + \Delta^{-1}\mathrm{d}s^2(\mathcal{M}_{SU(2)}), \tag{1.5}$$

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where $\mathcal{M}_{SU(2)}$ is of $SU(2)$ holonomy, is independent of the coordinates of $\mathcal{M}_{SU(3)}$, and $\mathcal{M}_{SU(3)}$ admits an $SU(3)$ structure. Again this is in contrast to the behaviour observed for fivebranes, though the effect is not so pronounced as for four-folds.

Finally, for membranes wrapped in a two-fold, we will see from the probe brane analysis that the structure group of the backgound is preserved in the presence of the probe. Then, in the supergravity description, we will see that the almost product structure of the spacetime is preserved, as is the local flatness of the overall transverse space, up to a warping. So in this case, the supersymmetry of the configuration protects the geometry of the overall transverse space from the gravitational effects of the membrane to the same degree as that observed for fivebranes; the metric in the supergravity description is given by

$$\mathrm{d}s^2 = -\Delta^2\mathrm{d}t^2 + \mathrm{d}s^2(\mathcal{M}_{SU(2)}) + \Delta^{-1}\mathrm{d}s^2(\mathbb{R}^6), \tag{1.6}$$

where $\mathcal{M}_{SU(2)}$ admits an $SU(2)$ structure.

We have found that the remaining supersymmetry conditions for wrapped membranes may be expressed in a very simple universal form. For fivefolds and fourfolds, in the supergravity regime the systems are indistinguishable, and the supersymmetry conditions are the same; in terms of the holomorphic five-form Ω and the almost complex structure J which specify the $SU(5)$ structure, they may be expressed as

$$\begin{aligned} \mathrm{d}(e^0 \wedge \mathrm{Re}\Omega) &= 0, \\ \mathrm{d} \star J &= 0, \\ F &= -\mathrm{d}(e^0 \wedge J). \end{aligned} \tag{1.7}$$

In the first of these equations we clearly recognise the generalised calibration condition for a probe M5 brane on a SLAG cycle in the backreacted geometry. In fact, this condition also implies $\mathrm{d}(e^0 \wedge \mathrm{Im}\Omega) = 0$, and of course a SLAG cycle may be calibrated by either the real or imaginary part of the holomorphic form. We also observe that the generalised calibration for the membrane worldvolume is required, by the four-form field equation, to be a harmonic form in spacetime. For threefolds or twofolds, if we let $\Delta^{n/2}I_n$, $n = 2, 3$, denote an arbitrary closed n-form on the overall transverse space, the supersymmetry conditions we find may be expressed as

$$\begin{aligned} \mathrm{d}(e^0 \wedge \Omega_{SU(n)} \wedge I_n) &= 0, \\ \mathrm{d} \star J_{SU(n)} &= 0, \\ F &= -\mathrm{d}(e^0 \wedge J_{SU(n)}). \end{aligned} \tag{1.8}$$

Once again, the first of these equations is manifestly a generalised calibration condition, for all the ways in which a probe fivebrane can wrap the backreacted geometry while preserving supersymmetry. And again the four-form field equation implies that the generalised calibration for the membrane worldvolume is harmonic in spacetime.

Once we have derived the wrapped brane supersymmetry conditions, we study their AdS_2 limits. In taking the AdS limit of a metric which is the warped product of a timelike line with a Riemannian ten-manifold, we must pick out the AdS radial direction from the ten-manifold. How we do this will be discussed in detail. For branes wrapped in a fivefold, the limiting procedure is completely general. Therefore our AdS_2 limit of the SLAG-5 supersymmetry conditions gives a classification of the general minimally supersymmetric AdS_2 spacetime in M-theory; from the AdS_2 limit of membranes on a holomorphic curve, we obtain a classification of the general minimally supersymmetric AdS_2 spacetime in M-theory with purely electric fluxes. Our supersymmetry conditions in this case coincide with those of [43], though we derive them without assuming that the nine-manifold transverse to the AdS factor is compact.

For membranes wrapped in fourfolds, there is nothing new to discuss in taking the AdS limit, since the wrapped brane supersymmetry conditions are identical to those for fivefolds. But for threefolds and twofolds, what we have found confirms and extends the beautiful relationship between bubbling geometries and AdS spaces proposed by LLM [41], and also makes manifest the intimate link between these geometries and the $SU(n)$ structures of the brane configurations. For membranes wrapped in a twofold, our limiting procedure produces a set of supersymmetry conditions for half-BPS AdS spacetimes. In addition to the AdS isometries, supersymmetry implies that these spacetimes will have $U(1) \times SO(6)$ isometry. The $SO(6)$ comes from the sphere in the overall transverse space, whose isometries are promoted to isometries of the full solution in our limit. The $U(1)$ arises because the AdS radial direction comes partly from the overall transverse space and partly from $\mathcal{M}_{SU(2)}$ in the wrapped brane metric; the part that lies in $\mathcal{M}_{SU(2)}$ is paired with a vector $\hat{w}$, which generates the $U(1)$, by $J_{SU(2)}$. Analytically continuing the metric and supersymmetry conditions to AdS_5, or to the bubbling geometries where the $U(1)$ is timelike, our conditions map exactly to those of LLM. The AdS_5 conditions of LLM were derived in [42] by taking the AdS limit of the supergravity description of fivebranes wrapped on holomorphic curves in twofolds.

For membranes wrapped in threefolds, we study the AdS limit when the overall transverse $SU(2)$ manifold is chosen to be $\mathbb{R}^4$. We find there are two ways of taking the AdS limit. One is when the AdS radial direction is assumed to lie entirely in the overall transverse space. In this case, the limit is $AdS_2 \times CY_3 \times S^3$. The AdS_3 limit of fivebranes wrapped on Kähler four-cycles in threefolds, again when the AdS radial direction is assumed to lie entirely in the overall transverse space of the wrapped brane metric, is $AdS_3 \times CY_3 \times S^2$. So again the supersymmetry conditions for the AdS limit of the membrane configuration analytically continue to those of the fivebrane configuration. But in this case there is no additional $U(1)$ isometry, so a bubbling interpretation is unclear. However, there is another $U(1)$ in the second way of taking the AdS limit of the wrapped membranes, where the AdS radial direction is assumed to lie partly in the overall transverse space and partly in the $SU(3)$ structure manifold of the wrapped brane metric. It has exactly the same origin as in the half-BPS case: it is the vector paired with the part of the AdS radial direction lying in $\mathcal{M}_{SU(3)}$ by $J_{SU(3)}$. Similarly, in [42] it was found that there is an extra $U(1)$ isometry in the AdS limit of fivebranes on four-cycles in threefolds, when the AdS radial direction comes partly from the threefold and partly from the overall transverse space. And we have found that the conditions we derive here for the AdS limit of the membranes analytically continue precisely to those of [42] for the AdS limit of the fivebranes. But now these spacetimes admit another analytic continuation; in Riemannian signature the isometry is $U(1) \times SO(3) \times SO(4)$, and continuing so that the $U(1)$ is timelike, we get spacetimes that we interpret as the quarter-BPS bubbling geometries of M-theory. We are unaware of any explicit known solutions of the supersymmetry conditions in this case, but it will be very interesting to study the supersymmetry conditions in more detail.

At this point it is worth briefly reviewing the literature on the brane configurations we study, and also supersymmetric AdS_2 spacetimes in M-theory. A gauged supergravity investigation of the near-horizon limits in some special cases was given for fivebranes on SLAG five-cycles in [29], and for membranes on holomorphic curves in [44]. The supersymmetry conditions for a single timelike Killing spinor in eleven dimensions (as appropriate for the description of M5 branes on SLAG five-cycles) were first given in [13]. Membranes wrapping holomorphic curves in Calabi-Yau manifolds were studied, using the Fayyazuddin-Smith ansatz and from the perspective of generalised calibrations, in [33] and [45]. The conditions for minimally supersymmetric AdS_2 spacetimes in M-theory, with vanishing magnetic flux and compact internal space, were derived in [43]. And of course the supersymmetry conditions for a class of half-BPS AdS_2 spacetimes in M-theory may be derived by analytic continuation of the results of LLM [41].

In this paper, we have not attempted to find any explicit new solutions of the supersymmetry conditions, and of course, performing classifications along the lines of those given here is only the first step in exploring the space of AdS/CFT duals in M-theory. To provide new explicit concrete examples of the duality, or new explicit bubbling solutions, the supersymmetry conditions (and the Bianchi identity/ field equations, where appropriate) must be solved on the supergravity side. However, the geometrical insight provided by the G-structure formalism has proven extremely useful in integrating the supersymmetry conditions. The by-now celebrated $Y^{p,q}$ spaces were constructed directly from the results of the AdS_5 classification of [39]; the field theory duals have been identified [46] and much further progress has been made. Some properties of the duals of other explicit AdS_5 solutions found in [39] have also very recently been elucidated [47]. Furthermore, inspired by the insight provided by the results of G-structure classifications, an extremely rich five-parameter family of supersymmetric AdS_3 solutions of M-theory has recently been constructed in [48]. These solutions are dual to field theories with $\mathcal{N} = (2, 0)$ supersymmetry, and arise as the near-horizon limits of M5 branes wrapping Kähler four-cycles in Calabi-Yau four-folds. Eight doubly-countably infinite compact families of these solutions, which may be dimensionally reduced and T-dualised to IIB, were studied in [49], and the central charges of the CFT duals computed. It is to be hoped that completing the classification of wrapped brane configurations and their AdS limits in M-theory may facilitate similar progress in the future.

The plan of the rest of this paper is as follows. In section 2, we study branes wrapping supersymmetric cycles in Calabi-Yau five-folds. From the supergravity description of fivebranes wrapping SLAG five-cycles together with membranes wrapping Kähler two-cycles, we derive the supersymmetry conditions for a general minimally supersymmetric AdS_2 spacetime in M-theory. We also show how the supersymmetry conditions for a minimally

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supersymmetric AdS_2 spacetime with purely electric fluxes [43] may be obtained directly from the supersymmetry conditions for membranes wrapped on a Kähler two-cycle.

In section 3, we study membranes wrapping Kähler two-cycles in Calabi-Yau n-folds, $n = 2, 3, 4$, performing a probe brane analysis and then deriving the supergravity description of the wrapped brane configurations.

In section 4, we study the AdS limits of these configurations. We describe our limiting procedure in detail, and employ it to derive the AdS supersymmetry conditions from those of the wrapped branes.

Section 5 concludes. Miscellaneous technical material is relegated from the main body of the text to several appendices. Throughout the text we use all the spinorial conventions of [13].

2. Branes wrapped on cycles in $SU(5)$ manifolds

In this section, we will study the supergravity description of branes wrapped on cycles in Calabi-Yau five-folds, together with their near-horizon limits. We are interested in two configurations: fivebranes wrapped on SLAG five-cycles in the presence of membranes on holomorphic curves; and secondly, configurations just with wrapped membranes. We will perform a brief probe-brane analysis, to identify the Killing spinors preserved in each case. Then we will discuss the supersymmetry conditions for the wrapped branes, from which we will finally derive the supersymmetry conditions for the AdS limits. Throughout this paper, all Killing spinors are timelike. We will therefore use a timelike spacetime basis, given by

$$ds^2 = -(e^0)^2 + \delta_{ab} e^a e^b, \tag{2.1}$$

where $a, b = 1, .., 9, \sharp$, in the following.

2.1 Probe branes

We are interested in probe branes on supersymmetric cycles in $\mathbb{R} \times CY_5$. We may choose the Killing spinors preserved by this background to be the pair of $SU(5)$ invariant spinors satisfying

$$\Gamma^{1234}\eta = \Gamma^{3456}\eta = \Gamma^{5678}\eta = \Gamma^{789\sharp}\eta = -\eta. \tag{2.2}$$

From the spinor bi-linears, we may construct the complex structure J and also the holomorphic five-form Ω, given by

$$\begin{aligned} J &= e^{12} + e^{34} + e^{56} + e^{78} + e^{9\sharp}, \\ \Omega &= (e^1 + ie^2)(e^3 + ie^4)(e^5 + ie^6)(e^7 + ie^8)(e^9 + ie^\sharp). \end{aligned} \tag{2.3}$$

Observe that the special holonomy projections imply that we can wrap a membrane for free on a Kähler two-cycle, calibrated by J, in the ten-manifold. We can take the membrane kappa-symmetry projection to be

$$\Gamma^{012}\eta = -\eta. \tag{2.4}$$

Therefore the two Killing spinors preserved by a membrane wrapping a holomorphic curve define an $SU(5)$ structure with two supersymmetries. Now consider introducing probe M5 branes wrapped on supersymmetric SLAG five-cycles. Such cycles are, by definition, calibrated by $\mathrm{Re}\Omega$. For such a cycle, we choose the projection

$$\Gamma^{013579}\eta = -\eta. \tag{2.5}$$

This projects out one of the Killing spinors of the CY_5, so these backgrounds preserve a single supersymmetry. We will reserve the notation ξ for a timelike spinor satisfying the projections (2.2) and (2.5). The second $SU(5)$ Killing spinor, projected out by (2.5), is

$$\frac{1}{10}J_{ab}\Gamma^{ab}\xi = \Gamma^0\xi. \tag{2.6}$$

Taking ξ to have unit norm, the forms it defines in eleven dimensions are

$$\begin{aligned} K &= \bar{\xi}\Gamma^{(1)}\xi = -e^0, \\ \Theta &= \bar{\xi}\Gamma^{(2)}\xi = J, \\ \Sigma &= \bar{\xi}\Gamma^{(5)}\xi = \frac{1}{2}e^0 \wedge J \wedge J + \mathrm{Re}\Omega. \end{aligned} \tag{2.7}$$

2.2 The supergravity description

Demanding the existence of Killing spinors defining the same algebraic structures as for the probe branes, and which are simultaneous eigenspinors of the projection operators of (2.2), (2.5), it is now an easy matter to give the supersymmetry conditions in the supergravity description. In going away from the probe brane approximation, the backreaction of the branes will deform the Calabi-Yau away from $SU(5)$ holonomy, but will still preserve $SU(5)$ structure. Let us now briefly state our bosonic ansatz. We demand that our wrapped brane metric is of the form

$$\mathrm{d}s^2 = -\Delta^2\mathrm{d}t^2 + h_{MN}\mathrm{d}x^M\mathrm{d}x^N. \tag{2.8}$$

Our timelike frame is realised by $e^0 = \Delta\mathrm{d}t$ and $e^a = e^a_M\mathrm{d}x^M$, where we refer to the ten-manifold spanned by e^a as the base B. Since we are interested in wrapped brane configurations admitting AdS limits, we have required that the timelike direction is not fibred over the base. The t-independence of Δ and h follows from the supersymmetry conditions of [13]. Our ansatz for the flux is

$$F = \Delta^{-1}e^0 \wedge H + G, \tag{2.9}$$

where H is a three-form and G is a four-form defined on B. We demand that H and G are independent of t. Now we give the supersymmetry conditions.

2.2.1 Fivebranes on SLAG five-cycles

In this case with a single timelike spinor, the supersymmetry conditions may be obtained simply by truncating the results of [13] to our ansatz. Turning on backreaction induces a warping of the timelike direction, and the forms defined by the Killing spinor will rescale

$$K = -\Delta e^0,$$

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$$\Theta = \Delta J,$$
$$\Sigma = \Delta\Big(\frac{1}{2}e^0 \wedge J \wedge J + \mathrm{Re}\Omega\Big). \tag{2.10}$$

The only restriction on the base $SU(5)$ structure implied by supersymmetry is

$$\mathrm{Re}\Omega \lrcorner \mathrm{dRe}\Omega = 8\mathrm{d}\log\Delta. \tag{2.11}$$

The four-form field strength is then given by

$$\begin{aligned} F &= -\mathrm{d}(e^0 \wedge J) + \frac{1}{2}\star\mathrm{d}(e^0 \wedge \mathrm{Re}\Omega) - \frac{1}{2}W_1 \wedge J + \frac{1}{4}(\mathrm{d}\log\Delta + W_4)\lrcorner\mathrm{Im}\Omega \\ &\quad + F^{\mathbf{75}} \\ &= -\mathrm{d}(e^0 \wedge J) - \frac{1}{2}W_2 - \frac{1}{3}W_1 \wedge J + \frac{1}{4}(\mathrm{d}\log\Delta + W_4)\lrcorner\mathrm{Im}\Omega + F^{\mathbf{75}}, \end{aligned} \tag{2.12}$$

where $F^{\mathbf{75}}$ is a four-form defined on the base in the **75** of $SU(5)$ which is unfixed by the supersymmetry conditions. The $SU(5)$ torsion modules $\mathcal{W}_i$, $i = 1, \ldots, 5$, are defined by

$$\mathrm{d}J = \frac{1}{8}\mathcal{W}_1\lrcorner\mathrm{Im}\Omega + \mathcal{W}_3 + \frac{1}{4}\mathcal{W}_4 \wedge J,$$
$$\mathrm{dRe}\Omega = \frac{1}{6}\mathcal{W}_1 \wedge J^2 + \mathcal{W}_2 \wedge J + \frac{1}{8}\mathrm{Re}\Omega \wedge \mathcal{W}_5. \tag{2.13}$$

At this point, we should clarify a potentially misleading aspect of our terminology. We refer to the supersymmetry conditions given above as "wrapped brane supersymmetry conditions"; however, they are sufficiently general that in addition to wrapped brane configurations, they are solved by many supersymmetric spacetimes which do not contain any branes at all. A trivial example is flat space with vanishing flux. On the other hand, the wrapped brane supersymmetry conditions will indeed be solved by all wrapped brane configurations in M-theory which admit an AdS_2 limit. With this understanding - that what we refer to as wrapped brane supersymmetry conditions are solved by all wrapped brane configurations in the desired class, but also admit other solutions - we will continue to use this terminology throughout the paper. And of course, none of the solutions to the wrapped brane supersymmetry conditions which do not in fact describe brane configurations will admit an AdS limit.

2.2.2 Membranes on holomorphic curves

The supersymmetry conditions for membranes wrapped on holomorphic curves may be derived simply by setting the magnetic flux to zero in (2.12). With the metric (2.8), the resulting equations may be succinctly expressed in the form given in the introduction,

$$\mathrm{d}(e^0 \wedge \mathrm{Re}\Omega) = 0,$$
$$\mathrm{d}\star J = 0,$$
$$F = -\mathrm{d}(e^0 \wedge J). \tag{2.14}$$

To verify that these configurations indeed admit two supersymmetries, observe that given an $SU(5)$ structure defined by (2.10) satisfying (2.14), we may always define a second

$SU(5)$ structure, satisfying (2.14), according to

$$\begin{aligned} K' &= \Delta e^0, \\ \Theta' &= -\Delta J, \\ \Sigma' &= \Delta\Big(-\frac{1}{2}e^0 \wedge J \wedge J + \mathrm{Re}\Omega\Big). \end{aligned} \tag{2.15}$$

These are the forms associated to the spinor $\Delta^{1/2}\Gamma^0\xi$, which is thus Killing.

2.3 The AdS limits

In this subsection, we will take the general AdS limits of the wrapped brane metrics and supersymmetry conditions of the previous subsections. To do so, we observe that a warped AdS_2 product metric may be viewed as a special case of the metric (2.8), if we write the AdS metric in Poincaré co-ordinates:

$$\frac{1}{m^2}\mathrm{d}s^2(AdS_2) = -e^{-2mr}\mathrm{d}t^2 + \mathrm{d}r^2. \tag{2.16}$$

Therefore to make contact with (2.8), we require that

$$\Delta = e^{-mr}\lambda^{-1/2}. \tag{2.17}$$

We demand that the AdS warp factor λ, and the frame on the space transverse to the AdS factor, are independent of the AdS coordinates. To complete the AdS metric ansatz, we must pick out the AdS radial direction from the base space. Using the transitive action of $SU(5)$ on the base, we may choose

$$\lambda^{-1/2}\mathrm{d}r = e^{\sharp}. \tag{2.18}$$

Picking out a preferred direction on the base associated to the doubling of supersymmetry reduces the structure group defined by the Killing spinors to $SU(4)$; the metric becomes

$$\mathrm{d}s^2 = \frac{1}{\lambda m^2}\mathrm{d}s^2(AdS_2) + \mathrm{d}s^2(\mathcal{M}_8) + e^9 \otimes e^9. \tag{2.19}$$

The $SU(4)$ structure is defined on $\mathcal{M}_8$. In terms of the $SU(4)$ structure forms,

$$\begin{aligned} J_{SU(4)} &= e^{12} + e^{34} + e^{56} + e^{78}, \\ \Omega_{SU(4)} &= (e^1 + ie^2)(e^3 + ie^4)(e^5 + ie^6)(e^7 + ie^8), \end{aligned} \tag{2.20}$$

the $SU(5)$ structure decomposes according to

$$\begin{aligned} J &= J_{SU(4)} + \lambda^{-1/2}e^9 \wedge \mathrm{d}r, \\ \mathrm{Re}\Omega &= \mathrm{Re}\Omega_{SU(4)} \wedge e^9 - \lambda^{-1/2}\mathrm{Im}\Omega_{SU(4)} \wedge \mathrm{d}r, \\ \mathrm{Im}\Omega &= \mathrm{Im}\Omega_{SU(4)} \wedge e^9 + \lambda^{-1/2}\mathrm{Re}\Omega_{SU(4)} \wedge \mathrm{d}r. \end{aligned} \tag{2.21}$$

To complete the AdS limit, we demand that the only non-vanishing electric flux contains a factor proportional to the AdS volume form, and that the magnetic flux has no components along the AdS radial direction.

2.3.1 The AdS limit of fivebranes on SLAG five-cycles

In this subsection, we will describe the result of this limiting procedure as applied to the SLAG five-cycle; a more detailed discussion of their derivation is given in appendix A. In the AdS limit there is an $SU(4)$ structure in nine dimensions; the forms defining this structure are e^9, $J_{SU(4)}$ and $\mathrm{Im}\Omega_{SU(4)}$ (we could instead have chosen $\mathrm{Re}\Omega_{SU(4)}$, their exterior derivatives contain the same information). For the remainder of this section, we will drop the $SU(4)$ subscripts; it is understood that all forms and torsion modules are now of $SU(4)$. The conditions on the intrinsic torsion, in terms of the $SU(4)$ torsion modules in nine dimensions, may be expressed as

$$\begin{aligned} \mathrm{d}(\lambda^{-1/2}J) &= 0, \\ \mathrm{dIm}\Omega &= \mathcal{W}_2^{(\mathbf{20}+\bar{\mathbf{20}})} \wedge J + \frac{1}{4}\mathrm{Im}\Omega \wedge \mathcal{W}_5^{(\mathbf{4}+\bar{\mathbf{4}})} \\ &\quad + \left[\lambda^{1/2} m \mathrm{Re}\Omega + \mathrm{Im}\Omega \partial_9 \log\lambda + \mathcal{W}_6^{(\mathbf{10}+\bar{\mathbf{10}})}\right] \wedge e^9, \\ \mathrm{d}e^9 &= \mathcal{W}_7^{(\mathbf{6}+\bar{\mathbf{6}})} + \mathcal{W}_8^{\mathbf{15}} + \frac{1}{4}\mathcal{W}_9^{\mathbf{1}} J + \frac{1}{2}(3\tilde{\mathrm{d}}\log\lambda - \mathcal{W}_5^{(\mathbf{4}+\bar{\mathbf{4}})}) \wedge e^9. \end{aligned} \tag{2.22}$$

The numbering of the torsion modules is chosen to emphasise that many modules are zero, and of those non-zero, many are not generic. We use $\tilde{\mathrm{d}}$ to denote the exterior derivative restricted to $\mathcal{M}_8$. The flux is given in terms of the torsion modules by

$$\begin{aligned} F = &\frac{1}{m^2}\mathrm{Vol}_{AdS_2} \wedge [\mathrm{d}(\lambda^{-1}e^9) - m\lambda^{-1/2}J] - \frac{1}{2}[\mathcal{W}_6 + (\mathcal{W}_7 \lrcorner \mathrm{Re}\Omega) \wedge J] \\ &+ \frac{1}{4}(\mathcal{W}_9 - m\lambda^{1/2})\mathrm{Re}\Omega + \frac{3}{8}\mathrm{Im}\Omega\partial_9 \log\lambda + F^{\mathbf{20}} \\ &- \left[J \cdot \mathcal{W}_2 + \frac{1}{4}(\mathcal{W}_5 - 4\tilde{\mathrm{d}}\log\lambda) \lrcorner \mathrm{Im}\Omega\right] \wedge e^9, \end{aligned} \tag{2.23}$$

where for an n-from Λ, we have defined $J \cdot \Lambda_{i_1 \dots i_n} = nJ_{[i_1}{}^{j}\Lambda_{|j|i_2 \dots i_n]}$, and $F^{\mathbf{20}}$ is a primitive $(2,2)$ form on $\mathcal{M}_8$ which is unfixed by the supersymmetry conditions.

2.3.2 The AdS limit of membranes on holomorphic curves

Now we will state the supersymmetry conditions we have derived for the AdS limit of membranes wrapping holomorphic curves; more details of their derivation are given in appendix A. There are two equivalent ways in which these supersymmetry conditions may be arrived at; first, by taking the AdS limit of (2.14) directly, and second, by setting the magnetic flux to zero in the AdS limit of the SLAG-5 supersymmetry conditions, (2.22), (2.23). It serves as a useful consistency check to verify that in commuting the order of these limits one indeed arrives at the same answer, and we have done so. The metric is given by

$$ds^2 = \frac{1}{\lambda m^2}[ds^2(AdS_2) + \lambda^{3/2} ds^2(\mathcal{M}_8) + (dz + \sigma)^2], \tag{2.24}$$

where ∂_z is Killing and $\mathcal{M}_8$ is Kähler, with Ricci form $\mathcal{R}$ and scalar curvature R given by

$$\mathcal{R} = \mathrm{d}\sigma, \tag{2.25}$$

$$R = 2\lambda^{3/2}. \tag{2.26}$$

The flux is given by

$$F = \mathrm{Vol}_{AdS_2} \wedge [\mathrm{d}(\lambda^{-1} e^9) - m\lambda^{-1/2} J_{SU(4)}]. \tag{2.27}$$

These conditions are identical to those of [43], though they are also valid for non-compact $\mathcal{M}_8$.

3. Membranes wrapped on cycles in $SU(n)$ manifolds

In this section, we will study the supergravity description of membranes wrapping holomorphic curves in Calabi-Yau four, three and two-folds. As discussed in the introduction, what we will see is that supersymmetric wrapped membranes can affect the geometry of spacetime in a way that is qualitatively different to that of wrapped fivebranes.

As in the previous section, we will first perform a probe brane analysis of the wrapped membrane configurations to determine the preserved supersymmetries, and the G-structures associated to them, and then derive the supergravity description of the same configurations with the same supersymmetries. In performing the probe brane and subsequent supergravity analysis, it is very useful to construct an explicit spinorial basis which respects $SU(5)$ covariance. This is one of the essential points of the refined G-structure formalism of [8, 15] — decomposing the space of spinors into modules of the structure group, and in so doing exploiting the geometrical structure present to organise and render tractable extremely complicated supergravity calculations. Using the timelike spacetime basis and the fiducial timelike spinor ξ defined in section 2, we choose our spinorial basis to be

$$\xi, \;\; \Gamma^0\xi, \;\; \Gamma^a\xi, \;\; \frac{1}{4}A^{(\mathbf{10}+\bar{\mathbf{10}})}_{ab}\Gamma^{ab}\xi. \tag{3.1}$$

Here $a, b = 1, \dots, 10$, and the $A^{(\mathbf{10}+\bar{\mathbf{10}})}$ furnish a basis for $(2,0)+(0,2)$ forms of $SU(5)$; explicitly, we may choose the $A^{(\mathbf{10}+\bar{\mathbf{10}})}$ to be $e^{13}-e^{24}$, $e^{14}+e^{23}$, etc. In choosing this basis, we are exploiting the isomorphism between the space of Majorana spinors in eleven dimensions and forms of $SU(5)$:

$$\mathbf{32} = (\mathbf{1}+\bar{\mathbf{1}}) \oplus (\mathbf{5}+\bar{\mathbf{5}}) \oplus (\mathbf{10}+\bar{\mathbf{10}}). \tag{3.2}$$

Observe that each member of this basis may be distinguished by its eigenvalues under the projection operators of (2.2) and (2.5). We will use this basis extensively in the following.

3.1 Probe brane analysis

3.1.1 Probe membranes on holomorphic curves in fourfolds

First we look at probe membranes wrapping a holomorphic curve in a fourfold. The background is $\mathbb{R}^{1,2} \times CY_4$, where we take the $\mathbb{R}^{1,2}$ to be spanned by e^0, e^9, and $e^\sharp$. The probe membrane is extended along the timelike direction e^0, and e^9, $e^\sharp$ span the overall transverse space. In the absence of the probe membrane, the background preserves four Killing spinors, which we may take to satisfy the projections

$$\Gamma^{1234}\eta = \Gamma^{3456}\eta = \Gamma^{5678}\eta = -\eta. \tag{3.3}$$

We may choose the four linearly independent solutions of these projection conditions to be

$$\xi,\ \Gamma^0\xi,\ \Gamma^9\xi,\ \Gamma^\sharp\xi. \tag{3.4}$$

These Killing spinors define an $SU(4)$ structure in eleven dimensions. The introduction of a membrane probe into this background breaks half of these supersymmetries. We must impose the kappa-symmetry projection for the probe brane, which we may choose to be

$$\Gamma^{012}\eta = -\eta. \tag{3.5}$$

This projects out the Killing spinors $\Gamma^9\xi,\ \ \Gamma^\sharp\xi$; the Killing spinors preserved by the background in the presence of the probe brane are

$$\xi,\ \ \Gamma^0\xi. \tag{3.6}$$

These Killing spinors define an $SU(5)$ structure; equivalently, they are annihilated by an $SU(5)$ subalgebra of the Lie algebra of $Spin(1,10)$. Thus we arrive at the surprising conclusion that wrapping a probe membrane in a Calabi-Yau fourfold, in breaking half the supersymmetries, increases the structure group defined by the Killing spinors from $SU(4)$ to $SU(5)$.

3.1.2 Probe membranes on holomorphic curves in threefolds

Now we look at probe membranes wrapped in a threefold. The background is $\mathbb{R}^{1,4} \times CY_3$, where now the overall transverse space is spanned by $e^7, \ldots, e^\sharp$. In the absence of the probe brane, the Killing spinors preserved by the background satisfy the projections

$$\Gamma^{1234}\eta = \Gamma^{3456}\eta = -\eta, \tag{3.7}$$

and we may choose them to be

$$\xi,\ \ \Gamma^0\xi,\ \ \Gamma^7\xi,\ \ \Gamma^8\xi,\ \ \Gamma^9\xi,\ \ \Gamma^\sharp\xi,\ \ \frac{1}{4}A^1_{ab}\Gamma^{ab}\xi,\ \ \frac{1}{4}A^2_{ab}\Gamma^{ab}\xi, \tag{3.8}$$

where

$$A^1 = e^{79} - e^{8\sharp},\ \ A^2 = e^{7\sharp} + e^{89}. \tag{3.9}$$

These eight Killing spinors define an $SU(3)$ structure. Now, introducing the membrane probe, we may again choose the kappa-symmetry projection to be (3.5); this projects out the four $\Gamma^a\xi$, $a = 7, \ldots, \sharp$, Killing spinors above. The surviving Killing spinors define an $SU(3) \times SU(2)$ structure in eleven dimensions; the most general element of the Lie algebra of $Spin(1,10)$ which annihilates all four is

$$B^{\mathbf{8}}_{ab}\Gamma^{ab} + C^{\mathbf{3}}_{ab}\Gamma^{ab}, \tag{3.10}$$

where $B^{\mathbf{8}}$ is an arbitrary primitive (1,1) form (ie, in the adjoint) of an $SU(3)$ acting on the 123456 directions, and $C^{\mathbf{3}}$ is an arbitrary primitive (1,1) form of an $SU(2)$ acting on the 789♯. Once again we observe the feature that in breaking half the supersymmetry, a probe membrane increases the structure group of the background.

3.1.3 Probe membranes on holomorphic curves in twofolds

Finally, we look at membranes wrapped in a twofold. In this case, the background is $\mathbb{R}^{1,6} \times CY_2$, and we take the overall transverse space to be spanned by $e^5, \ldots, e^\sharp$. The sixteen Killing spinors of the background satisfy the single projection

$$\Gamma^{1234}\eta = -\eta, \tag{3.11}$$

and the sixteen basis spinors satisfying this projection may be easily found. Introducing the probe brane, we must once again impose the kappa-symmetry projection, which we again choose to be (3.5). The eight surviving Killing spinors are

$$\xi, \;\; \Gamma^0\xi, \;\; A^A_{ab}\Gamma^{ab}\xi, \;\; A = 1, \ldots, 6, \tag{3.12}$$

where $A^{1,2}$ are as defined in (3.9), and

$$\begin{aligned} A^3 &= e^{59} - e^{6\sharp}, \;\; A^4 = e^{5\sharp} + e^{69}, \\ A^5 &= e^{57} - e^{68}, \;\; A^6 = e^{58} + e^{67}. \end{aligned} \tag{3.13}$$

The most general element of $Spin(1,10)$ annihilating these eight Killing spinors is

$$D^{\mathbf{3}}_{ab}\Gamma^{ab}, \tag{3.14}$$

where $D^{\mathbf{3}}$ is a primitive (1,1) form of an $SU(2)$ acting on the 1234 directions. Therefore these Killing spinors define an $SU(2)$ structure in eleven dimensions. In this case, the structure group of the background is preserved under the introduction of the probe brane.

3.2 The supergravity description

Now we will use the Killing spinors obtained in the probe brane approximation for the fermionic part of the supergravity ansatz. We will demand that the Killing spinors for the supergravity description lie in a subbundle of the spin bundle spanned by the Killing spinors of the probe brane description. We will further assume that the supergravity Killing spinors are, up to multiplication by arbitrary functions, the same as the probe brane Killing spinors given above. This second assumption is the same as saying that we assume the supergravity Killing spinors to be simultaneous eigenspinors of the five projection operators of (2.2) and (2.5)[1]. For the bosonic part of the ansatz for the supergravity description, we demand that the spacetime is a warped product of a timelike line with a ten-manifold, so the metric is of the form of (2.8):

$$\mathrm{d}s^2 = -\Delta^2\mathrm{d}t^2 + \mathrm{d}s^2(\mathcal{M}_{10}). \tag{3.15}$$

To complete our bosonic ansatz we demand that the magnetic flux vanishes. This exhausts our assumptions in deriving the supergravity description.

[1] This second assumption may in fact be redundant. For all of the cases involving M5 branes studied in [42] for which this was checked, orthogonality of the supergravity Killing spinors was in fact implied by the Killing spinor equation. However, we have not checked this for the case in hand.

Supersymmetry implies that the warp factor Δ and the frame on $\mathcal{M}_{10}$ are independent of t, and furthermore that Δ is related algebraically in the same way to the norms of the individual Killing spinors, so the *a priori* arbitrary functions we allow for in the Killing spinors must be the same. The remainder of the supersymmetry conditions, which in the rest of this subsection we give for each case in turn, will restrict the geometry of $\mathcal{M}_{10}$ and the form of the electric flux.

3.2.1 Membranes wrapped in fourfolds: the supergravity description

In this case, the probe brane analysis revealed that the Killing spinors preserved by the system define an $SU(5)$ structure, and are algebraically identical to those for a membrane wrapped in a five-fold. Given our assumptions for the supergravity description, we have in fact already worked out the supersymmetry conditions for this case; they are identical to those given for membranes wrapped in a fivefold.

In particular, any information regarding the original $SU(4)$ structure of the background is lost in the supergravity description. As discussed in the introduction, we interpret this to mean that the almost product structure of the spacetime is not protected by supersymmetry in going to the supergravity regime.

3.2.2 Membranes wrapped in threefolds: the supergravity description

In this case, the probe brane Killing spinors define an $SU(3) \times SU(2)$ structure. We take the supergravity Killing spinors to be

$$\Delta^{1/2}\xi, \;\; \Delta^{1/2}\Gamma^0\xi, \;\; \Delta^{1/2}\frac{1}{4}A^1_{ab}\Gamma^{ab}\xi, \;\; \Delta^{1/2}\frac{1}{4}A^2_{ab}\Gamma^{ab}\xi, \tag{3.16}$$

where now nothing is assumed about the frame e^a on $\mathcal{M}_{10}$ beyond its t-independence. To derive the supersymmetry conditions with these Killing spinors, we first impose that $\Delta^{1/2}\xi$ and $\Delta^{1/2}\Gamma^0\xi$ solve the Killing spinor equation with our bosonic ansatz. Since together these spinors define an $SU(5)$ structure, they once again imply the supersymmetry conditions of subsection 2.2.2: $\mathcal{M}_{10}$ admits an $SU(5)$ structure, and the torsion conditions and flux are given by

$$\begin{aligned} \mathrm{d}(e^0 \wedge \mathrm{Re}\Omega) &= 0, \\ \mathrm{d} \star J &= 0, \\ F &= -\mathrm{d}(e^0 \wedge J). \end{aligned} \tag{3.17}$$

Now we must impose the conditions implied by the existence of the additional pair $\eta_{(1)} = \Delta^{1/2}\frac{1}{4}A^1_{ab}\Gamma^{ab}\xi$, $\eta_{(2)} = \Delta^{1/2}\frac{1}{4}A^2_{ab}\Gamma^{ab}\xi$. Observing that $\Gamma^0\eta_{(1)} = \eta_{(2)}$, we see that $\eta_{(1)}$ and $\eta_{(2)}$ collectively define a different $SU(5)$ structure. Therefore their existence must imply the existence of a second solution of (3.17), but with different structure forms $(e^0)'$, J', $\mathrm{Re}\Omega'$. These forms may be computed from the bilinears of $\eta_{(1)}$, and we find

$$\begin{aligned} (e^0)' &= e^0, \\ J' &= J_{SU(3)} - J_{SU(2)}, \end{aligned}$$

$$\mathrm{Re}\Omega' = \mathrm{Re}\Omega_{SU(3)} \wedge \mathrm{Re}\Omega_{SU(2)} + \mathrm{Im}\Omega_{SU(3)} \wedge \mathrm{Im}\Omega_{SU(2)}, \tag{3.18}$$

where we have defined

$$\begin{aligned} J_{SU(3)} &= e^{12} + e^{34} + e^{56}, \\ \Omega_{SU(3)} &= (e^1 + ie^2)(e^3 + ie^4)(e^5 + ie^6), \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} J_{SU(2)} &= e^{78} + e^{9\sharp}, \\ \Omega_{SU(2)} &= (e^7 + ie^8)(e^9 + ie^\sharp). \end{aligned} \tag{3.20}$$

By contrast, the $SU(5)$ forms defined by the Killing spinor $\Delta^{1/2}\xi$ decompose under $SU(3)\times SU(2)$ according to

$$\begin{aligned} J &= J_{SU(3)} + J_{SU(2)}, \\ \mathrm{Re}\Omega &= \mathrm{Re}\Omega_{SU(3)} \wedge \mathrm{Re}\Omega_{SU(2)} - \mathrm{Im}\Omega_{SU(3)} \wedge \mathrm{Im}\Omega_{SU(2)}. \end{aligned} \tag{3.21}$$

Thus, on demanding that both the primed and the unprimed forms satisfy (3.17), we find the necessary and sufficient conditions for the existence of the four Killing spinors (3.16) given our bosonic ansatz. These conditions are

$$\begin{aligned} \mathrm{d}(\Delta\mathrm{Re}\Omega_{SU(3)} \wedge \mathrm{Re}\Omega_{SU(2)}) &= 0, \\ \mathrm{d}(\Delta\mathrm{Im}\Omega_{SU(3)} \wedge \mathrm{Im}\Omega_{SU(2)}) &= 0, \\ \mathrm{d}\star J_{SU(3)} = \mathrm{d}\star J_{SU(2)} &= 0, \\ \mathrm{d}(\Delta J_{SU(2)}) &= 0, \\ F &= -\mathrm{d}(e^0 \wedge J_{SU(3)}). \end{aligned} \tag{3.22}$$

In appendix B, we analyse these conditions in detail, and we find that they may be considerably simplified. To state them, we first define a quaternionic two-form $I_{SU(2)}$, in terms of the unit quaternions (i, j, k) and the invariant $SU(2)$ forms:

$$I_{SU(2)} = iJ_{SU(2)} + j\mathrm{Re}\Omega_{SU(2)} + k\mathrm{Im}\Omega_{SU(2)}. \tag{3.23}$$

We find that (3.22) implies that

$$\mathrm{d}(\Delta I_{SU(2)}) = 0. \tag{3.24}$$

Therefore the spacetime admits an almost product structure; if we conformally rescale the metric along the $SU(2)$ directions according to

$$\mathrm{d}s^2 = -\Delta^2\mathrm{d}t^2 + \mathrm{d}s^2(\mathcal{M}_{SU(3)}) + \Delta^{-1}\mathrm{d}s^2(\mathcal{M}_{SU(2)}), \tag{3.25}$$

then (3.24) implies that $\mathcal{M}_{SU(2)}$ has $SU(2)$ holonomy, and the frame on it may be chosen to be independent of the coordinates, and the coordinate differentials, of $\mathcal{M}_{SU(3)}$. The remaining conditions constrain the $SU(3)$ structure on $\mathcal{M}_{SU(3)}$, together with determining the flux. We find that they reduce to

$$\begin{aligned} \mathrm{d}(e^0 \wedge \Omega_{SU(3)} \wedge I_{SU(2)}) &= 0, \\ \mathrm{d} \star J_{SU(3)} &= 0, \\ F &= -\mathrm{d}(e^0 \wedge J_{SU(3)}). \end{aligned} \tag{3.26}$$

The formal similarity of these conditions with those for $SU(5)$ is striking. From the first of these conditions we may read off all the ways in which a probe M5 may be wrapped in the backreacted geometry, while preserving supersymmetry: on the product of a SLAG three cycle in $\mathcal{M}_{SU(3)}$ (caibrated by either the real or imaginary parts of $\Omega_{SU(3)}$) with a holomorphic curve (which could be calibrated by $J_{SU(2)}$, $\mathrm{Re}\Omega_{SU(2)}$ or $\mathrm{Im}\Omega_{SU(2)}$) in the $SU(2)$ manifold.

3.2.3 Membranes wrapped in twofolds: the supergravity description

Now we turn to the supersymmetry conditions for membranes wrapped in two-folds. The derivation proceeds in a very similar way to that for threefolds. The most convenient way to obtain the conditions is to observe that the $SU(2)$ structure defined by the Killing spinors in this case is equivalent to a pair of $SU(3) \times SU(2)$ structures, and to use the supersymmetry conditions for each. We will just state the result. We find that the metric is given by

$$\mathrm{d}s^2 = -\Delta^2 \mathrm{d}t^2 + \mathrm{d}s^2(\mathcal{M}_{SU(2)}) + \Delta^{-1}\mathrm{d}s^2(\mathbb{R}^6). \tag{3.27}$$

Thus in this case, the flatness of the overall transverse is protected by supersymmetry to the same extent as it is for fivebranes. The supersymmetry conditions fit the by-now familiar pattern; if $\Delta^{3/2}\Lambda$ is an arbitrary closed three-form on the overall transverse space, the supersymmetry conditions may be expressed as

$$\mathrm{d}(e^0 \wedge \Omega_{SU(2)} \wedge \Lambda) = 0, \tag{3.28}$$

$$\mathrm{d} \star J_{SU(2)} = 0, \tag{3.29}$$

$$F = -\mathrm{d}(e^0 \wedge J_{SU(2)}), \tag{3.30}$$

where here

$$J_{SU(2)} = e^{12} + e^{34}, \tag{3.31}$$

$$\Omega_{SU(2)} = (e^1 + ie^2)(e^3 + ie^4). \tag{3.32}$$

Observe that (3.28) is equivalent to $\mathrm{d}(\Delta^{-1/2}\Omega_{SU(2)}) = 0$. Again, this is just a generalised calibration condition for an M5 probe in the backreacted geometry. Observe however that there is no condition of the form $\mathrm{d}(e^0 \wedge J_{SU(2)} \wedge \Lambda) = 0$. In an $SU(2)$ holonomy manifold, there is essentially no distinction between holomorphic curves calibrated by J or the real or imaginary parts of Ω. However because we have picked one of the complex structures to calibrate the cycle wrapped by the membranes, and then included backreaction, the symmetry of the complex structures is broken, and so there is no $e^0 \wedge J_{SU(2)} \wedge \Lambda$ generalised calibration for probe fivebranes.

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4. The AdS limits of wrapped membranes

In this section, we study the AdS limits, and associated supersymmetry conditions, of the supergravity description of membranes wrapped in threefolds and twofolds. As in section 2.3, this involves making a suitable ansatz for the warp factor and frame, picking the AdS radial direction out of the ten Riemannian dimensions, and imposing vanishing of the flux components not containing the AdS volume form as a factor. Picking out the AdS radial direction is now somewhat less trivial, because the wrapped brane metrics in these cases are

$$\mathrm{d}s^2 = -\Delta^2\mathrm{d}t^2 + \mathrm{d}s^2(\mathcal{M}_{10-p}) + \Delta^{-1}\mathrm{d}s^2(\mathcal{N}_p), \tag{4.1}$$

where $\mathcal{N}_p$ is an $SU(2)$ holonomy manifold for membranes wrapped in a threefold and $\mathcal{N}_p = \mathbb{R}^6$ for a twofold. For the case of a threefold, we will restrict attention to the special case $\mathcal{N}_p = \mathbb{R}^4$ henceforth. It would be interesting to know if there are other choices for the $SU(2)$ manifold that admit an AdS limit, but we think this is unlikely and we will not pursue this question here. Thus the wrapped brane metrics we study are

$$\mathrm{d}s^2 = -\Delta^2\mathrm{d}t^2 + \mathrm{d}s^2(\mathcal{M}_{9-q}) + \Delta^{-1}(\mathrm{d}R^2 + R^2\mathrm{d}s^2(S^q)), \tag{4.2}$$

where $q = 3$ for threefolds and $q = 5$ for twofolds. Generically, the AdS radial direction will lie partly in $\mathcal{M}_{9-q}$ and partly in the overall transverse space. It cannot lie entirely in $\mathcal{M}_{9-q}$, because then explicit dependence on the AdS radial coordinate will enter through the Δ^{-1} warp factor of the overall transverse space, in contradiction of our assumption of a warped AdS product. However we will see that for threefolds (but not for twofolds) the AdS radial direction can lie entirely in the overall transverse space. This non-generic case will be discussed seperately below, but here we will focus on describing our limiting procedure in the generic case, where the AdS radial direction lies partly in $\mathcal{M}_{9-q}$ and partly in the overall transverse space. Our treatment very closely follows that of [42]. We will assume that the part of the AdS radial direction lying in the overall transverse lies entirely along the radial direction of the overall transverse space. We emphasise that this is indeed an assumption, and in contrast to the $SU(5)$ case, we are no longer gauranteed that we are taking the most general AdS limit of the wrapped brane configurations (though we believe that in fact we are).

We may extract the AdS radial direction by performing a frame rotation. Defining, as in section 2.3,

$$\Delta = e^{-mr}\lambda^{-1/2}, \tag{4.3}$$

where r is the AdS radial coordinate, we write, for some one-form $\hat{u}$ lying entirely in $\mathcal{M}_{9-q}$ and for $\hat{v} = \Delta^{-1/2}\mathrm{d}R$,

$$\lambda^{-1/2}\mathrm{d}r = \sin\theta\hat{u} + \cos\theta\hat{v}, \tag{4.4}$$

where we assume that the rotation angle θ is independent of r. We also define the orthogonal combination

$$\hat{\rho} = \cos\theta\hat{u} - \sin\theta\hat{v}. \tag{4.5}$$

Inverting these expressions to give $\hat{v}$ in terms of the new frame, then imposing closure of $\mathrm{d}R$, we find that

$$\hat{\rho} = \frac{2\lambda^{1/4}}{m\sin\theta}\mathrm{d}(\lambda^{-3/4}\cos\theta). \tag{4.6}$$

Defining a new coordinate $\rho = \lambda^{-3/4}\cos\theta$, we get

$$\begin{aligned}
\hat{\rho} &= \frac{2\lambda^{1/4}}{m\sqrt{1-\lambda^{3/2}\rho^2}}\mathrm{d}\rho, \\
\hat{u} &= \lambda^{-1/2}\sqrt{1-\lambda^{3/2}\rho^2}\mathrm{d}r + \frac{2\lambda\rho\mathrm{d}\rho}{m\sqrt{1-\lambda^{3/2}\rho^2}}, \\
R &= -\frac{2}{m}\rho e^{-mr/2}.
\end{aligned} \tag{4.7}$$

Therefore, the spacetime metric in this AdS limit becomes

$$\mathrm{d}s^2 = \frac{1}{\lambda m^2}\Big(\mathrm{d}s^2(AdS_2) + 4\lambda^{3/2}\Big[\frac{\mathrm{d}\rho^2}{1-\lambda^{3/2}\rho^2} + \rho^2\mathrm{d}s^2(S^q)\Big]\Big) + \mathrm{d}s^2(\mathcal{M}_{8-q}), \tag{4.8}$$

where $\mathrm{d}s^2(\mathcal{M}_{8-q})$ is defined by

$$\mathrm{d}s^2(\mathcal{M}_{9-q}) = \mathrm{d}s^2(\mathcal{M}_{8-q}) + \hat{u}\otimes\hat{u}. \tag{4.9}$$

In taking this AdS limit, we are picking out preferred vectors (associated to the additional AdS Killing spinors) in both $\mathcal{M}_{9-q}$ and the overall transverse space. This will reduce the structure group of the wrapped brane spacetime: from $SU(3)\times SU(2)$ to an $SU(2)$ subgroup of the $SU(3)$ factor for threefolds, and from $SU(2)$ to the identity for twofolds. To complete the AdS limit, we impose vanishing of the flux components not containing a factor of the AdS volume form.

4.1 The AdS limit of membranes on threefolds

As we have mentioned, in addition to the more generic limiting procedure discussed above, the supergravity description for membranes wrapped in a threefold admits a special AdS limit, where the AdS radial direction is assumed to lie entirely in the overall transverse space. We will first discuss this special case, before moving on to the more generic one.

4.1.1 AdS radial direction from the overall transverse space

Demanding that the AdS radial coordinate lies entirely in the overall transverse space implies that λ is constant (and may be set to unity by rescaling) and that $\partial/\partial r$ lies along the radial direction of the overall transverse space. Since in this case no preferred vector associated to a Killing spinor is picked out on the $SU(3)$ manifold, the structure group of the AdS limit is reduced to $SU(3)$ here. The metric is given by

$$\mathrm{d}s^2 = \frac{1}{m^2}[\mathrm{d}s^2(AdS_2) + 4\mathrm{d}s^2(S^3)] + \mathrm{d}s^2(\mathcal{M}_{SU(3)}). \tag{4.10}$$

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Then the supersymmetry conditions imply that

$$\mathrm{d}J_{SU(3)} = \mathrm{d}\Omega_{SU(3)} = 0, \tag{4.11}$$

and hence the spacetime is the direct product $AdS_2 \times S^3 \times CY_3$. The flux is given by $F = -\mathrm{Vol}_{AdS_2} \wedge J_{SU(3)}/m$. This solution is of course well known. For fivebranes wrapped on Kähler four-cycles, it is also possible to take a non-generic AdS limit of the wrapped brane supersymmetry conditions, in exactly the same way as is done here [42]. The non-generic fivebrane AdS_3 limit turns out to be the analytic continuation of the non-generic membrane AdS_2 limit.

4.2 Generic case

The AdS supersymmetry conditions in the more generic case, where the AdS radial direction is assumed to lie partly in the $SU(3)$ structure manifold and partly along the radial direction of the overall transverse space, are worked out in detail in appendix C. The metric is given by

$$\begin{aligned} \mathrm{d}s^2 = & \frac{1}{\lambda m^2}\Big(\mathrm{d}s^2(AdS_2) + 4\lambda^{3/2}\Big[\frac{\mathrm{d}\rho^2}{1-\lambda^{3/2}\rho^2} + \rho^2\mathrm{d}s^2(S^3)\Big]\Big) \\ & + \mathrm{d}s^2(\mathcal{M}_{SU(2)}) + w \otimes w, \end{aligned} \tag{4.12}$$

where $\mathcal{M}_{SU(2)}$ admits an $SU(2)$ structure. If we define the $SU(2)$ structure forms

$$\begin{aligned} J^1 &= e^{12} + e^{34}, \\ J^2 &= e^{14} + e^{23}, \\ J^3 &= e^{13} - e^{24}, \end{aligned} \tag{4.13}$$

the conditions on the intrinsic torsion are

$$\mathrm{d}\left(\lambda^{-1/2}\sqrt{1-\lambda^{3/2}\rho^2}J^2\right) = 0, \tag{4.14}$$

$$\mathrm{d}\left(\lambda^{-1/2}\sqrt{1-\lambda^{3/2}\rho^2}J^3\right) = 0, \tag{4.15}$$

$$\mathrm{d}(\lambda^{-1/2}J^1 + \lambda^{1/4}\rho\hat{w} \wedge \hat{\rho}) = 0, \tag{4.16}$$

$$J^3 \wedge \mathrm{d}\left(\frac{\lambda^{1/2}}{\sqrt{1-\lambda^{3/2}\rho^2}}\hat{w}\right) = J^2 \wedge \mathrm{d}\left(\frac{1}{\lambda^{1/4}\rho\sqrt{1-\lambda^{3/2}\rho^2}}\hat{\rho}\right), \tag{4.17}$$

$$J^2 \wedge \mathrm{d}\left(\frac{\lambda^{1/2}}{\sqrt{1-\lambda^{3/2}\rho^2}}\hat{w}\right) = -J^3 \wedge \mathrm{d}\left(\frac{1}{\lambda^{1/4}\rho\sqrt{1-\lambda^{3/2}\rho^2}}\hat{\rho}\right). \tag{4.18}$$

The flux is given by

$$F = \frac{1}{m^2}\mathrm{Vol}_{AdS_2} \wedge \left[\mathrm{d}\left(\lambda^{-1}\sqrt{1-\lambda^{3/2}\rho^2}\hat{w}\right) - m(\lambda^{-1/2}J^1 + \lambda^{1/4}\rho\hat{w} \wedge \hat{\rho})\right]. \tag{4.19}$$

As discussed in the introduction, in the generic case also, the supersymmetry conditions we obtain are the analytic continuation of the conditions of [42] for the generic AdS_3 limit of M5s wrapped on a Kähler four-cycle in a threefold. For the M5s, the isometry of the generic AdS_3 limit is $SO(3) \times U(1)$; for the membranes, the equations given above imply that the isometry of the generic AdS_2 limit is $SO(4) \times U(1)$, with the $SO(4)$ coming from the sphere and the $U(1)$ generated by $\hat{w}$. We may also analytically continue so that the $U(1)$ isometry becomes the timelike direction, and so obtain supersymmetric M-theory spacetimes with $SO(4) \times SO(3)$ isometry. We identify the spacetimes satisfying these supersymmetry conditions as 1/4 BPS bubbling solutions in M-theory.

4.3 The AdS limit of membranes on twofolds

Now we will give the conditions we derive for the AdS limit of membranes on a twofold. Observe in that in this case, $\mathrm{d}(\Delta^{-1/2}\Omega_{SU(2)}) = 0$, it is not possible to take a non-generic AdS limit, with the AdS radial direction lying entirely in the overall transverse space. The derivation of the supersymmetry conditions is very similar to that for threefolds in the generic case, and technical details are omitted. The metric is given by

$$\begin{aligned} \mathrm{d}s^2 = & \frac{1}{\lambda m^2}\Big(\mathrm{d}s^2(AdS_2) + 4\lambda^{3/2}\Big[\frac{\mathrm{d}\rho^2}{1-\lambda^{3/2}\rho^2} + \rho^2 \mathrm{d}s^2(S^5)\Big]\Big) \\ & + e^1 \otimes e^1 + e^2 \otimes e^2 + e^3 \otimes e^3, \end{aligned} \tag{4.20}$$

with $\hat{w} = e^3$. The torsion conditions we derive are

$$\mathrm{d}\left(\lambda^{-1/4}\sqrt{1-\lambda^{3/2}\rho^2}\,e^1\right) = -\frac{m\lambda^{1/4}}{2}\left(\lambda^{3/4}\rho e^1 \wedge \hat{\rho} + e^{23}\right), \tag{4.21}$$

$$\mathrm{d}\left(\lambda^{-1/4}\sqrt{1-\lambda^{3/2}\rho^2}\,e^2\right) = -\frac{m\lambda^{1/4}}{2}\left(\lambda^{3/4}\rho e^2 \wedge \hat{\rho} - e^{13}\right), \tag{4.22}$$

$$\begin{aligned} \mathrm{d}\left(\frac{\lambda^{1/2}}{\sqrt{1-\lambda^{3/2}\rho^2}}e^3\right) = & -\frac{2m\lambda}{1-\lambda^{3/2}\rho^2}e^{12} - \frac{3\lambda^{1/4}\rho}{2(1-\lambda^{3/2}\rho^2)^{3/2}}(\partial_{\hat{\rho}}\lambda e^{12} \\ & -\partial_2\lambda e^1 \wedge \hat{\rho} + \partial_1\lambda e^2 \wedge \hat{\rho}). \end{aligned} \tag{4.23}$$

The flux is then given by

$$F = \frac{1}{m^2}\mathrm{Vol}_{AdS_2} \wedge \left[\mathrm{d}\left(\lambda^{-1}\sqrt{1-\lambda^{3/2}\rho^2}\,e^3\right) - m\lambda^{-1/2}\left(e^{12} + \lambda^{3/4}\rho e^3 \wedge \hat{\rho}\right)\right]. \tag{4.24}$$

Upon analytic continuation, these give precisely the conditions of LLM [41], expressed in a form similar to that of [42].

5. Conclusions

In this work, we have performed a systematic study of the geometry of wrapped brane configurations admitting AdS_2 limits in eleven dimensional supergravity. We have found that the backreaction of wrapped membranes (in threefolds or fourfolds) on their overall transverse space is less restricted by the kinematics of the Killing spinor equation than that hitherto observed for fivebranes on any cycle. The reason for this can ultimately be traced back to the Killing spinors. For the cycles wrapped by fivebranes studied in [42], it was found that the Killing spinors preserved in the presence of the brane are isomorphic to vectors of $Spin(7)$ (recall that the isotropy group of a null spinor in eleven dimensions is $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$; for the wrapped fivebrane configurations of [42], the Killing spinors are null). These vectors essentially lock the frame on the overall transverse space in passing from the probe brane to the supergravity description, and it is their presence which ultimately restricts the gravitational effect of the fivebrane on the overall transverse space. By contrast, the Killing spinors preserved by probe membranes are isomorphic to zero or two-forms of $SU(5)$; the vectors are projected out by the kappa-symmetry projections. Thus (for fourfolds and threefolds) one cannot define preferred vectors on the overall transverse space associated to the supersymmetries, and this is ultimately the kinematical origin of the effect we have observed.

We have seen how the supersymmetry conditions for wrapped membranes in Calabi-Yau n-folds may be expressed in a simple universal form. We have also made manifest the link, via analytic continuation, between the AdS limits of the supergravity description of membranes and fivebranes wrapped in threefolds or twofolds, and 1/4 and 1/2 BPS bubbling solutions in M-theory.

It will be interesting to examine these 1/4-BPS bubbling geometries in more detail, and in particular, to try to find some explicit solutions of the supersymmetry conditions (assuming, of course, that some exist) together with their field theory duals. In the 1/2-BPS case, different boundary conditions must be imposed in the different (AdS_2, AdS_5 or bubbling) branches in order to get regular solutions; it will be interesting to see if something similar applies here.

It will also be interesting to apply the techniques of [42] to the study of fivebranes wrapping four-cycles in eight-manifolds. This case is particularly rich, and there are many possibilities to consider. One can include membranes, intersecting the fivebranes in a string, and extended in the directions transverse to the eight-manifold. The new explicit AdS_3 solutions of [48, 49] come from this sector of M-theory. A general analysis of the different possible cases is under investigation [51].

6. Acknowledgements

OC is supported by EPSRC. We are grateful to Pau Figueras for initial collaboration on this project, and to Jerome Gauntlett for useful discussions and comments.

A. Branes wrapping cycles in five-folds: technical details

In this appendix, we will give the technical details of the derivation of the supersymmetry conditions for the near-horizon limits of branes wrapping cycles in $SU(5)$ manifolds.

A.1 The AdS limit of fivebranes on SLAG five-cycles

To derive the AdS supersymmetry conditions, we first decompose the $SU(5)$ modules of the wrapped brane structure group into modules of the $SU(4)$ structure group of the near-horizon limit, under which the metric decomposes according to

$$ds^2(\mathcal{M}_{10}) = ds^2(\mathcal{M}_8) + e^9 \otimes e^9 + \lambda^{-1}\mathrm{d}r^2, \tag{A.1}$$

with $\mathcal{M}_8$ admitting an $SU(4)$ structure. We start with the flux term in the **75** of $SU(5)$. Under $SU(4)$, this decomposes as

$$\mathbf{75} \to \mathbf{20} + \mathbf{15} + (\mathbf{20} + \bar{\mathbf{20}}). \tag{A.2}$$

The flux term in the **75** of $SU(5)$ thus decomposes according to

$$\begin{aligned} F^{\mathbf{75}} &= F_1^{\mathbf{20}} + F_2^{\mathbf{15}} \wedge (J_{SU(4)} - 2e^{9\sharp}) + F_3^{(\mathbf{20}+\bar{\mathbf{20}})} \wedge e^9 \\ &- J_{SU(4)} \cdot F_3^{(\mathbf{20}+\bar{\mathbf{20}})} \wedge e^\sharp, \end{aligned} \tag{A.3}$$

where for an n-from Λ, we have defined $J \cdot \Lambda_{i_1 \ldots i_n} = nJ_{[i_1}{}^{j}\Lambda_{|j|i_2\ldots i_n]}$, and we have used $J_{SU(5)} \lrcorner F^{\mathbf{75}} = 0$.

Next, we decompose the $SU(5)$ torsion modules into modules of $SU(4)$. Since $(\mathbf{10} + \bar{\mathbf{10}}) \to (\mathbf{6} + \bar{\mathbf{6}}) + (\mathbf{4} + \bar{\mathbf{4}})$, W_1 decomposes according to

$$W_1^{(\mathbf{10}+\bar{\mathbf{10}})} = A_1^{(\mathbf{6}+\bar{\mathbf{6}})} + A_2^{(\mathbf{4}+\bar{\mathbf{4}})} \wedge e^9 + J_{SU(4)} \cdot A_2^{(\mathbf{4}+\bar{\mathbf{4}})} \wedge e^\sharp. \tag{A.4}$$

For W_2, since $(\mathbf{40} + \bar{\mathbf{40}}) \to (\mathbf{20} + \bar{\mathbf{20}}) + (\mathbf{10} + \bar{\mathbf{10}}) + (\mathbf{6} + \bar{\mathbf{6}}) + (\mathbf{4} + \bar{\mathbf{4}})$, we find that

$$\begin{aligned} W_2^{(\mathbf{40}+\bar{\mathbf{40}})} &= B_1^{(\mathbf{10}+\bar{\mathbf{10}})} + B_2^{(\mathbf{6}+\bar{\mathbf{6}})} \wedge (J_{SU(4)} - 2e^{9\sharp}) + (B_3^{(\mathbf{4}+\bar{\mathbf{4}})} \lrcorner \mathrm{Im}\Omega_{SU(4)} + B_4^{(\mathbf{20}+\bar{\mathbf{20}})}) \wedge e^9 \\ &+ (-B_3^{(\mathbf{4}+\bar{\mathbf{4}})} \lrcorner \mathrm{Re}\Omega_{SU(4)} + J_{SU(4)} \cdot B_4^{(\mathbf{20}+\bar{\mathbf{20}})}) \wedge e^\sharp. \end{aligned} \tag{A.5}$$

For W_3, since $(\mathbf{45}+\bar{\mathbf{45}}) \to (\mathbf{20}+\bar{\mathbf{20}})+(\mathbf{6}+\bar{\mathbf{6}})+(\mathbf{4}+\bar{\mathbf{4}})+\mathbf{15}+\mathbf{15'}$, we have the decomposition

$$\begin{aligned} W_3^{(\mathbf{45}+\bar{\mathbf{45}})} &= C_1^{(\mathbf{20}+\bar{\mathbf{20}})} + C_2^{(\mathbf{4}+\bar{\mathbf{4}})} \wedge (J_{SU(4)} - 3e^{9\sharp}) + (C_3^{(\mathbf{6}+\bar{\mathbf{6}})} + C_4^{\mathbf{15}}) \wedge e^9 \\ &+ \Big(-\frac{1}{2} J_{SU(4)} \cdot C_3^{(\mathbf{6}+\bar{\mathbf{6}})} + C_5^{\mathbf{15'}} \Big) \wedge e^\sharp. \end{aligned} \tag{A.6}$$

The modules W_4 and W_5 decompose as vectors, $(\mathbf{5} + \bar{\mathbf{5}}) \to (\mathbf{4} + \bar{\mathbf{4}}) + (\mathbf{1} + \bar{\mathbf{1}})$:

$$\begin{aligned} \mathcal{W}_4 &= D_1^{(\mathbf{4}+\bar{\mathbf{4}})} + D_2 e^9 + D_3 e^\sharp, \\ \mathcal{W}_5 &= E_1^{(\mathbf{4}+\bar{\mathbf{4}})} + E_2 e^9 + E_3 e^\sharp. \end{aligned} \tag{A.7}$$

Now to obtain the flux and torsion conditions in the AdS limit, we simply impose vanishing of the magnetic flux components along $e^\sharp$, the vanishing of electric flux components not containing a factor proportional to the AdS volume form, and also decompose both sides of equation (2.13) under $SU(4)$. Defining $Z = \mathrm{d}\log\Delta + \mathcal{W}_4$, imposing vanishing of the magnetic flux along the AdS radial direction, we find that

$$\begin{aligned} B_2^{(\mathbf{6}+\bar{\mathbf{6}})} &= \frac{1}{3} A_1^{(\mathbf{6}+\bar{\mathbf{6}})}, \\ A_2^{(\mathbf{4}+\bar{\mathbf{4}})} &= 0, \end{aligned}$$

$$B_3^{(\mathbf{4}+\bar{\mathbf{4}})} = -\frac{1}{2} Z^{(\mathbf{4}+\bar{\mathbf{4}})}. \tag{A.8}$$

The surviving magnetic flux is then given by

$$\begin{aligned} F_{mag} = &-\frac{1}{2}(B_1^{(\mathbf{10}+\bar{\mathbf{10}})} + 3B_2^{(\mathbf{6}+\bar{\mathbf{6}})} \wedge J_{SU(4)}) + \frac{1}{4}(Z_9 \mathrm{Im}\Omega_{SU(4)} + Z_\sharp \mathrm{Re}\Omega_{SU(4)}) \\ &- (B_4^{(\mathbf{20}+\bar{\mathbf{20}})} + B_3^{(\mathbf{4}+\bar{\mathbf{4}})} \lrcorner \mathrm{Im}\Omega_{SU(4)}) \wedge e^9. \end{aligned} \tag{A.9}$$

Next we decompose equations (2.13) under $SU(4)$. The equation for $\mathrm{d}J$ implies that

$$\begin{aligned} \mathrm{d}J_{SU(4)} &= \left(C_2^{(\mathbf{4}+\bar{\mathbf{4}})} + \frac{1}{4} D_1^{(\mathbf{4}+\bar{\mathbf{4}})}\right) \wedge J_{SU(4)} + C_1^{(\mathbf{20}+\bar{\mathbf{20}})} \\ &+ \left(\frac{3}{8} B_2^{(\mathbf{6}+\bar{\mathbf{6}})} \lrcorner \mathrm{Im}\Omega_{SU(4)} + C_3^{(\mathbf{6}+\bar{\mathbf{6}})} + C_4^{\mathbf{15}} + \frac{1}{4} D_2 J_{SU(4)}\right) \wedge e^9, \end{aligned} \tag{A.10}$$

$$\begin{aligned} \lambda^{1/2}\mathrm{d}(\lambda^{-1/2} e^9) &= \frac{3}{8} B_2^{(\mathbf{6}+\bar{\mathbf{6}})} \lrcorner \mathrm{Re}\Omega_{SU(4)} - \frac{1}{2} J_{SU(4)} \cdot C_3^{(\mathbf{6}+\bar{\mathbf{6}})} + C_5^{\mathbf{15}'} \\ &+ \frac{1}{4} D_3 J_{SU(4)} + \left(\frac{1}{4} D_1^{(\mathbf{4}+\bar{\mathbf{4}})} - 3C_2^{(\mathbf{4}+\bar{\mathbf{4}})}\right) \wedge e^9. \end{aligned} \tag{A.11}$$

From the equation for $\mathrm{dRe}\Omega$, we get

$$\begin{aligned} \lambda^{1/2}\mathrm{d}(\lambda^{-1/2}\mathrm{Im}\Omega_{SU(4)}) &= \left(B_3^{(\mathbf{4}+\bar{\mathbf{4}})} \lrcorner \mathrm{Re}\Omega_{SU(4)} - J_{SU(4)} \cdot B_4^{(\mathbf{20}+\bar{\mathbf{20}})}\right) \wedge J_{SU(4)} \\ -\frac{1}{8}\mathrm{Im}\Omega_{SU(4)} \wedge E_1^{(\mathbf{4}+\bar{\mathbf{4}})} &- \left[B_1^{(\mathbf{10}+\bar{\mathbf{10}})} + \frac{1}{8}\left(E_3 \mathrm{Re}\Omega_{SU(4)} + E_2 \mathrm{Im}\Omega_{SU(4)}\right)\right] \wedge e^9, \end{aligned} \tag{A.12}$$

$$\begin{aligned} \mathrm{d}(\mathrm{Re}\Omega_{SU(4)} \wedge e^9) = \frac{3}{2} B_2^{(\mathbf{6}+\bar{\mathbf{6}})} \wedge J^2_{SU(4)} &+ \left[\left(B_3^{(\mathbf{4}+\bar{\mathbf{4}})} \lrcorner \mathrm{Im}\Omega_{SU(4)} + B_4^{(\mathbf{20}+\bar{\mathbf{20}})}\right) \wedge J_{SU(4)} \right. \\ &\left. -\frac{1}{8}\mathrm{Re}\Omega_{SU(4)} \wedge E_1^{(\mathbf{4}+\bar{\mathbf{4}})}\right] \wedge e^9. \end{aligned} \tag{A.13}$$

Next, imposing that the electric flux contains a factor proportional to the AdS volume form, we get

$$\mathrm{d}(\lambda^{-1/2} J_{SU(4)}) = 0, \tag{A.14}$$

$$F_{elec} = \frac{1}{m^2}\mathrm{Vol}_{AdS_2} \wedge [\mathrm{d}(\lambda^{-1} e^9) - m\lambda^{-1/2} J_{SU(4)}]. \tag{A.15}$$

Finally we must impose the $SU(5)$ torsion condition, $\mathcal{W}_5 = 8\mathrm{d}\log\Delta$.

To make further progress, we first compare equation (A.10) with (A.14). This implies the following torsion conditions

$$\begin{aligned} C_1^{(\mathbf{20}+\bar{\mathbf{20}})} = C_4^{\mathbf{15}} &= 0, \\ B_2^{(\mathbf{6}+\bar{\mathbf{6}})} \lrcorner \mathrm{Im}\Omega_{SU(4)} &= -\frac{8}{3} C_3^{(\mathbf{6}+\bar{\mathbf{6}})}, \\ C_2^{(\mathbf{4}+\bar{\mathbf{4}})} + \frac{1}{4} D_1^{(\mathbf{4}+\bar{\mathbf{4}})} &= \frac{1}{2}\tilde{\mathrm{d}}\log\lambda, \\ D_2 &= 2\partial_9 \log\lambda, \end{aligned} \tag{A.16}$$

where here $\tilde{\mathrm{d}}$ denotes the exterior derivative restricted to $\mathcal{M}_8$. At this point, using the algebraic restrictions we have derived on the torsion, we may eliminate all vector and singlet modules in favour of $D_1^{(\mathbf{4}+\bar{\mathbf{4}})}$, D_3, and derivatives of λ. Upon doing this, we find that (A.13) is in fact implied by equations (A.11) and (A.12). We may thus summarise the supersymmetry conditions as

$$\begin{aligned}
\mathrm{d}(\lambda^{-1/2}J) &= 0, \\
\mathrm{d}e^9 &= \frac{3}{4}B_2^{(\mathbf{6}+\bar{\mathbf{6}})}\lrcorner\mathrm{Re}\Omega_{SU(4)} + C_5^{\mathbf{15}'} + \frac{1}{4}D_3 J + (D_1^{(\mathbf{4}+\bar{\mathbf{4}})} - \tilde{\mathrm{d}}\log\lambda)\wedge e^9, \\
\mathrm{dIm}\Omega_{SU(4)} &= -J_{SU(4)}\cdot B_4^{(\mathbf{20}+\bar{\mathbf{20}})}\wedge J_{SU(4)} + \mathrm{Im}\Omega_{SU(4)}\wedge\left(\frac{5}{4}\tilde{\mathrm{d}}\log\lambda - \frac{1}{2}D_1^{(\mathbf{4}+\bar{\mathbf{4}})}\right) \\
&\quad + \left[\lambda^{1/2}m\mathrm{Re}\Omega_{SU(4)} + \partial_9\log\lambda\mathrm{Im}\Omega_{SU(4)} - B_1^{(\mathbf{10}+\bar{\mathbf{10}})}\right]\wedge e^9, \\
F &= \frac{1}{m^2}\mathrm{Vol}_{AdS_2}\wedge[\mathrm{d}(\lambda^{-1}e^9) - m\lambda^{-1/2}J_{SU(4)}] + F_1^{\mathbf{20}} - \frac{1}{2}B_1^{(\mathbf{10}+\bar{\mathbf{10}})} \\
&\quad - \frac{3}{2}B_2^{(\mathbf{6}+\bar{\mathbf{6}})}\wedge J_{SU(4)} + \frac{3}{8}\partial_9\log\lambda\mathrm{Im}\Omega_{SU(4)} + \frac{1}{4}(D_3 - m\lambda^{1/2})\mathrm{Re}\Omega_{SU(4)} \\
&\quad - \left[B_4^{(\mathbf{20}+\bar{\mathbf{20}})} + \frac{1}{4}(\tilde{\mathrm{d}}\log\lambda - 2D_1^{(\mathbf{4}+\bar{\mathbf{4}})})\lrcorner\mathrm{Im}\Omega_{SU(4)}\right]\wedge e^9.
\end{aligned} \tag{A.17}$$

Relabelling the torsion modules we obtain the expressions quoted in the main text.

A.2 The AdS_2 limit of membranes on holomorphic curves

We will now give further technical details of the derivation of the AdS limit of the supersymmetry conditions for membranes on holomorphic curves in five-folds, (2.14). As discussed in the main text, there are two equivalent ways in which these conditions may be arrived at; though as it is quicker to obtain the result by setting the magnetic fluxes to zero in (2.23), this is what will be presented here. The metric in the AdS limit is

$$\mathrm{d}s^2 = \frac{1}{\lambda m^2}\mathrm{d}s^2(AdS_2) + \mathrm{d}s^2(\mathcal{M}_8) + e^9\otimes e^9, \tag{A.18}$$

with $\mathcal{M}_8$ admitting an $SU(4)$ structure. Throughout this subsection, all forms and modules are those of $SU(4)$.

Thus, setting the magnetic fluxes to zero in (2.23), the conditions (2.22) reduce to

$$\partial_9\lambda = 0, \tag{A.19}$$

$$\mathrm{d}(\lambda^{1/2}e^9) = \frac{1}{4}m\lambda J + \mathcal{W}^{\mathbf{15}}, \tag{A.20}$$

$$\mathrm{d}(\lambda^{-1/2}J) = 0, \tag{A.21}$$

$$\mathrm{d}(\lambda^{-1}\mathrm{Im}\Omega) = m\lambda^{-1/2}e^9\wedge\mathrm{Re}\Omega, \tag{A.22}$$

where $\mathcal{W}^{\mathbf{15}}$ is a two-form in the adjoint of $SU(4)$ which is unfixed by the supersymmetry conditions.

To make progress in solving these conditions, let us introduce coordinates z, x such that

$$e^9 = \frac{A}{m}(\mathrm{d}z + \sigma), \tag{A.23}$$

where σ is a one-form defined on the base $SU(4)$ manifold with coordinates x, and *a priori* A and σ depend on $z,\ x$. Now, we observe that (A.19) together with the 9 component of (A.21) implies that

$$\partial_z \lambda = \partial_z J = 0. \tag{A.24}$$

Next, the 9 component of (A.20) gives

$$\partial_z(\lambda^{1/2} A\sigma) = \tilde{\mathrm{d}}(\lambda^{1/2} A), \tag{A.25}$$

or

$$\sigma = \frac{1}{\lambda^{1/2} A}\Big(\tilde{\mathrm{d}} \int^z \lambda^{1/2} A \mathrm{d}z' + \sigma_0(x)\Big). \tag{A.26}$$

By choosing a new coordinate, we may always take $A = \lambda^{-1/2}$, $\sigma = \sigma(x)$; explicitly, we choose

$$z' = \int^z \lambda^{1/2} A \mathrm{d}z''. \tag{A.27}$$

Then $e^9 = m^{-1}\lambda^{-1/2}(\mathrm{d}z' + \sigma_0(x)) = m^{-1}e^{9'}$; dropping the primes and subscripts, this is the gauge in which we will work henceforth.

At this point, it is convenient to conformally rescale the base space, $g_8 = m^{-2}\lambda^{1/2}\hat{g}_8$, $J = m^{-2}\lambda^{1/2}\hat{J}$, $\Omega = m^{-4}\lambda\hat{\Omega}$. Henceforth we will work only with the rescaled base metric, and drop the hats. Then (A.22) becomes

$$\mathrm{dIm}\Omega = \lambda^{1/2}\mathrm{Re}\Omega \wedge e^9. \tag{A.28}$$

If we define

$$\Lambda_{ij} = (\partial_z e^i)_j, \tag{A.29}$$

then the z-independence of J together with the 9 component of (A.28) are equivalent to

$$\Lambda_{ij} = -\frac{1}{4} J_{ij} + \Lambda^{\mathbf{15}}_{ij}. \tag{A.30}$$

We may always eliminate the $\Lambda^{\mathbf{15}}$ term by performing a z-dependent $SU(4)$ rotation of the frame on the base. Thus we may solve for the z-dependence of the frame for the base space, according to

$$\begin{aligned} e^1 &= -\sin\frac{z}{4}\tilde{e}^2(x) + \cos\frac{z}{4}\tilde{e}^1(x), \\ e^2 &= \cos\frac{z}{4}\tilde{e}^2(x) + \sin\frac{z}{4}\tilde{e}^1(x), \end{aligned} \tag{A.31}$$

and similarly for the other pairs of basis one-forms. This rotation to the tilded frame leaves the metric and J invariant, but shifts Ω by a phase:

$$\Omega(z,x) = e^{iz}\tilde{\Omega}(x). \tag{A.32}$$

Then the remaining content of (A.28) may be expressed as

$$\mathrm{d}\tilde{\Omega} = i\sigma \wedge \tilde{\Omega}. \tag{A.33}$$

This, together with $dJ = 0$, implies that the conformally rescaled base admits a Kähler metric, with Ricci form

$$\mathcal{R} = d\sigma. \tag{A.34}$$

Observe that this is consistent with the absence of $(2,0)+(0,2)$ forms in (A.20). The final remaining condition comes from the singlet of (A.20). This is equivalent to the condition

$$R = 2\lambda^{3/2}, \tag{A.35}$$

where R is the scalar curvature of the base. These conditions, summarised in the main text, are precisely those of [43].

B. Membranes on $SU(n)$ manifolds: technical details

In this appendix, we will give more technical details of the supersymmetry conditions for membranes wrapped in threefolds. The derivation for twofolds is similar and is omitted.

In the main text, a set of supersymmetry conditions for the supergravity description of membranes wrapped in threefolds was derived. Manipulating these conditions into the final form subsequently quoted is a useful exercise in $SU(3) \times SU(2)$ structures. We start by conformally rescaling the $SU(2)$ directions according to

$$\mathrm{d}s^2 = -\Delta^2 \mathrm{d}t^2 + \mathrm{d}s^2(\mathcal{M}_{SU(3)}) + \Delta^{-1}\mathrm{d}s^2(\mathcal{M}_{SU(2)}). \tag{B.1}$$

Then in terms of the rescaled forms (which we use throughout this section) the equations of the main text become

$$\mathrm{d}(\mathrm{Re}\Omega_{SU(3)} \wedge \mathrm{Re}\Omega_{SU(2)}) = 0, \tag{B.2}$$
$$\mathrm{d}(\mathrm{Im}\Omega_{SU(3)} \wedge \mathrm{Im}\Omega_{SU(2)}) = 0, \tag{B.3}$$
$$\mathrm{d}J_{SU(2)} = 0, \tag{B.4}$$
$$J_{SU(2)} \wedge \mathrm{dVol}_{SU(3)} = 0, \tag{B.5}$$
$$\mathrm{Vol}_{SU(2)} \wedge \mathrm{d}(\Delta^{-1} J^2_{SU(3)}) = 0. \tag{B.6}$$

To analyse these equations, we introduce the following notation. Let upper-case letters $A, B, .. = 1, \ldots, 6$ denote the $SU(3)$ directions, and let lower-case letters $a, b = 7, \ldots, \sharp$ denote the $SU(2)$ directions. Let $\tilde{\mathrm{d}}$ denote the exterior derivative restricted to the 1,. . . ,6 directions, and let $\hat{\mathrm{d}}$ denote the exterior derivative restricted to the $7, \ldots, \sharp$ directions. We say a form is a $(p;q)$ form if it has p indices along the $SU(3)$ directions and q indices along the $SU(2)$ directions. Furthermore, let us define

$$\mathrm{d}e^A = \frac{1}{2} U^A_{ab} e^a \wedge e^b + V^{AB}_a e^a \wedge e^B + \tilde{\mathrm{d}}e^A, \tag{B.7}$$

$$\mathrm{d}e^a = \frac{1}{2} X^a_{AB} e^A \wedge e^B + Y^{ab}_A e^A \wedge e^b + \hat{\mathrm{d}} e^a. \tag{B.8}$$

Now we begin the analysis. First, observe that the only torsion module contained in (B.6) is (in standard notation) the $\mathcal{W}_4$ module of the $SU(3)$ structure on $\mathcal{M}_6$. Furthermore, this module appears nowhere else, so (B.6) is completely independent of the other equations. Next we look at (B.4). We may re-express this as

$$J_{SU(2)ab} X^a \wedge e^b + J_{SU(2)ab} Y^{ac} \wedge e^c \wedge e^b + \hat{\mathrm{d}} J_{SU(2)} = 0. \tag{B.9}$$

Since the first of the terms is a (2;1) form, the second a (1;2) form and the third a (0;3) form, they must all vanish seperately. Thus

$$\hat{\mathrm{d}} J_{SU(2)} = 0, \tag{B.10}$$
$$X^a = 0, \tag{B.11}$$
$$J_{SU(2)c[a} Y_{b]}^{\ \ c} = 0. \tag{B.12}$$

Observe that on contracting the third of these equations with $J_{SU(2)ab}$ we find that $Y^a_{\ a} = 0$. Next look at (B.5). This is equivalent to

$$J_{SU(2)} \wedge U^A = 0, \tag{B.13}$$
$$V^A_{\ \ A} = 0. \tag{B.14}$$

Equations (B.2) and (B.3) remain. The $(4;2)$ part of (B.2) gives

$$(\tilde{\mathrm{d}} \mathrm{Re}\Omega_{SU(3)} \mathrm{Re}\Omega_{SU(2)ab} + 2\mathrm{Re}\Omega_{SU(3)} \mathrm{Re}\Omega_{SU(2)c[a} Y^c_{\ \ b]}) \wedge e^a \wedge e^b = 0. \tag{B.15}$$

Contracting the term in parentheses with $\mathrm{Re}\Omega_{SU(2)ab}$, we find that

$$\tilde{\mathrm{d}} \mathrm{Re}\Omega_{SU(3)} = 0, \tag{B.16}$$
$$\mathrm{Re}\Omega_{SU(2)c[a} Y^c_{\ \ b]} = 0. \tag{B.17}$$

Similarly for the $(4;2)$ part of (B.3). Now, equations (B.12) and (B.17) imply that Y^{ab} must be antisymmetric, and, with orientation $\mathrm{Vol}_{SU(2)} = \frac{1}{2} J_{SU(2)} \wedge J_{SU(2)}$, anti-selfdual in the indices a, b. Therefore, regarded as the components of an $SU(2)$ two-form, the Y^{ab} lie in the adjoint, and this implies that they may be set to zero locally by performing an $SU(2)$ rotation of the 789♯ directions, while preserving the metric, $J_{SU(2)}$ and $\Omega_{SU(2)}$. Thus we have

$$\mathrm{d}e^a = \hat{\mathrm{d}} e^a. \tag{B.18}$$

Therefore we may always locally choose the frame on $\mathcal{M}_{SU(2)}$ to be independent of the coordinates, and coordinate differentials, of $\mathcal{M}_{SU(3)}$.

Next consider the $(2;4)$ parts of (B.2), (B.3). From these we find

$$\Omega_{SU(2)} \wedge U^A = 0. \tag{B.19}$$

Finally, from the (3;3) part of (B.2) we get

$$3\mathrm{Re}\Omega_{SU(3)D[AB}V^D{}_{C]}\mathrm{Re}\Omega_{SU(2)} - \mathrm{Re}\Omega_{SU(3)ABC}\hat{\mathrm{d}}\mathrm{Re}\Omega_{SU(2)} = 0. \tag{B.20}$$

Contracting this equation with $\mathrm{Re}\Omega_{SU(3)ABC}$, using (B.14), we find that

$$\mathrm{Re}\Omega_{SU(3)D[AB}V^D{}_{C]} = 0, \tag{B.21}$$
$$\hat{\mathrm{d}}\mathrm{Re}\Omega_{SU(2)} = 0. \tag{B.22}$$

Similarly for the (3; 3) part of (B.3).

We have now exhausted all the torsion conditions, so let us summarise what we have found. The conditions on the $SU(2)$ forms are

$$\mathrm{d}J_{SU(2)} = \mathrm{d}\Omega_{SU(2)} = 0. \tag{B.23}$$

With $I_{SU(2)}$ defined as in the main text, the conditions on the $SU(3)$ forms may be summarised as

$$\mathrm{Vol}_{SU(2)} \wedge \mathrm{d}(\Delta^{-1}J^2_{SU(3)}) = \mathrm{d}\star J_{SU(3)} = 0, \tag{B.24}$$
$$I_{SU(2)} \wedge U^A = 0, \tag{B.25}$$
$$\Omega_{SU(3)D[AB}V^D{}_{C]} = 0, \tag{B.26}$$
$$\tilde{\mathrm{d}}\Omega_{SU(3)} = 0. \tag{B.27}$$

It is readily verified that the last three equations may be combined into

$$I_{SU(2)} \wedge \mathrm{d}\Omega_{SU(3)} = 0. \tag{B.28}$$

Thus we obtain the results quoted in the main text.

C. The AdS limit of membranes on three folds and twofolds: technical details

In this appendix, we will give more of the technical details of the derivation of the AdS limit of the wrapped brane supersymmetry conditions for threefolds. The derivation for twofolds is very similar, and is omitted. Our starting point is the equations

$$\mathrm{d}(e^0 \wedge \Omega_{SU(3)} \wedge I_{SU(2)}) = 0, \tag{C.1}$$
$$\mathrm{d}\star J_{SU(3)} = 0, \tag{C.2}$$
$$F = -\mathrm{d}(e^0 \wedge J_{SU(3)}). \tag{C.3}$$

By using the transitive action of $SU(3)$ in six dimensions, we may choose the part of the AdS radial direction lying in $\mathcal{M}_{SU(3)}$ to lie along e^6; then we have $\hat{w} = e^5$. It is straightforward to derive (4.16) and (4.18) for the flux by demanding that the only non-vanishing components of (C.3) contain a factor proportional to the AdS volume form.

Next look at (C.2). After some manipulation, and using (4.16), this may be shown to be equivalent to

$$\mathrm{Vol}_{S^3} \wedge \mathrm{d}[\lambda^{-1}(1-\lambda^{3/2}\rho^2)\mathrm{Vol}_{\mathcal{M}_{SU(2)}}] = 0. \tag{C.4}$$

We will see that this condition will in fact be implied by one of the others we will derive. It remains to look at (C.1). We choose a basis for the self-dual $SU(2)$ forms on the overall transverse space according to

$$K^a = \mathrm{d}R \wedge \sigma^a + \frac{1}{4}\epsilon^{abc}\sigma^b \wedge \sigma^c, \tag{C.5}$$

where the σ^a are the $SU(2)$ invariant one-forms. Then the part of (C.1) containing the real part of $\Omega_{SU(3)}$ gives

$$\begin{aligned} &\mathrm{d}\left[J^3 \wedge \hat{w} - \frac{1}{\lambda^{3/4}\rho}J^2 \wedge \hat{\rho}\right] \wedge \sigma^a \wedge \mathrm{d}r \\ &-\mathrm{d}\left[\lambda^{-1}\sqrt{1-\lambda^{3/2}\rho^2}J^2\right] \wedge \frac{1}{2}\epsilon^{abc}\sigma^b \wedge \sigma^c \wedge \mathrm{d}r = 0, \end{aligned} \tag{C.6}$$

with a similar equation from the $\mathrm{Im}\Omega_{SU(3)}$ part. These equations do not at first sight obviously imply those given in the main text. However, observe that (4.16) implies that $(\sigma^a \wedge \sigma^b)\lrcorner\mathrm{d}J^1 = (\sigma^a \wedge \sigma^b)\lrcorner\mathrm{d}\hat{w} = 0$. This in turn implies $(\sigma^a \wedge \sigma^b)\lrcorner\mathrm{d}J^a = 0$. Then, wedging (C.6) with σ^a, we find

$$\mathrm{d}\left[\lambda^{-1}\sqrt{1-\lambda^{3/2}\rho^2}J^2\right] \wedge \mathrm{Vol}_{S^3} = 0, \tag{C.7}$$

and hence that

$$\mathrm{d}\left[\lambda^{-1}\sqrt{1-\lambda^{3/2}\rho^2}J^2\right] = \sigma^a \wedge \left(\sigma^a \lrcorner \mathrm{d}\left[\lambda^{-1}\sqrt{1-\lambda^{3/2}\rho^2}J^2\right]\right). \tag{C.8}$$

But then we may write (C.6) schematically as

$$A \wedge \sigma^a - B^a \wedge \mathrm{Vol}_{S^3} = 0, \tag{C.9}$$

and since $(\sigma^a \wedge \sigma^b)\lrcorner A = 0$, we have $A = 0$, $B^a = 0$, and therefore that

$$\begin{aligned} \mathrm{d}\left[\lambda^{-1}\sqrt{1-\lambda^{3/2}\rho^2}J^2\right] &= 0, \\ \mathrm{d}\left[J^3 \wedge \hat{w} - \frac{1}{\lambda^{3/4}\rho}J^2 \wedge \hat{\rho}\right] &= 0. \end{aligned} \tag{C.10}$$

The first of these equations implies (C.4), and so we obtain the results quoted in the main text.

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